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ON SEPARATING PREDICTABILITY AND DETERMINISM

ABSTRACT. There has been a long-standing debate about the relationship of predictability and determinism. Some have maintained that determinism implies predictability while others have maintained that predictability implies determinism. Many have maintained that there are no implication relations between determinism and predictability. This summary is, of course, somewhat oversimplified and quick at least in the sense that there are various notions of determinism and predictability at work in the philosophical literature. In this essay I will focus on what I take to be the Laplacean vision for determinism and predictability. While many forms of predictability are inconsistent with this vision, I argue that a suitably restricted notion of predictability, consistent with the practice of physicists, is implied by the Laplacean notion of determinism. It is argued that limitations on predictability are of an *in principle* nature in the Appendix.

We ought to regard the present state of the universe as the effect of its antecedent state and as the cause of the state that is to follow. An intelligence knowing all the forces acting in nature at a given instant, as well as the momentary positions of all things in the universe, would be able to comprehend in one single formula the motions of the largest bodies as well as the lightest atoms in the world, provided that its intellect were sufficiently powerful to subject all data to analysis; to it nothing would be uncertain, the future as well as the past would be present to its eyes. The perfection that the human mind has been able to give to astronomy affords but a feeble outline of such an intelligence. Discoveries in mechanics and geometry, coupled with those in universal gravitation, have brought the mind within reach of comprehending in the same analytical formula the past and the future state of the system of the world. All of the mind's efforts in the search for truth tend to approximate the intelligence we have just imagined, although it will forever remain infinitely remote from such an intelligence (Laplace as translated in Nagel 1961, pp. 281–282).

1. INTRODUCTION

The belief that any deterministic system is predictable has been part of our scientific traditions in some form from their beginnings through the twentieth century. This belief is persistent because of the power of the intuitions that lie behind the concept of physical determinism. The philosophical

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literature discussing determinism and predictability is quite diverse (e.g., Laplace 1951; Bunge 1967; Boyd 1972; Popper 1950; Earman 1986; Hunt 1987; Stone 1989; Van Kampen 1991; Batterman 1993; Kellert 1993; Bricmont 1995; Schurz 1996; Schmidt 1998b) and I do not intend to summarize the whole of the discussions in this essay. However, as a quick overview, some authors clearly specify that they have predictability for all times in mind (Popper 1950, p. 117; Feigl 1953, p. 411; Russell 1953, p. 400; Boyd 1972, p. 440; Schmidt 1998b; Hunt 1987), while others take predictability to be for finite times only (Popper 1950, pp. 123–127; Earman 1986, pp. 163–166; Van Kampen 1991, p. 276; Batterman 1993, pp. 52–54; Kellert 1993, p. 62; Bricmont 1995, p. 163; Smith 1998, p. 55). Furthermore, almost all authors take any limitations on predictability to be an in practice matter. Exceptions are Stephen Kellert (see footnote 14), Peter Smith (see Appendix A), Bertrand Russell (see footnote 10), Jan Schmidt (Section 4) and Mark Stone (Section 3).

This essay will focus on what I take to be Laplace's vision of physics – more precisely, classical point mechanics (CPM) – and explicate what, if any, form of predictability is implied in this vision. In Section 2 I will describe the vision and its motivations, while in Section 3 I will argue that a limited, but useful notion of predictability is implied by the deterministic core of the vision. Section 4 contains some concluding thoughts and Section 5 is an appendix discussing the nature of error sources in measurements and representations.

2. THE LAPLACEAN VISION

Stone (1989) has given a useful summary of the key elements constituting what I call the Laplacean vision for determinism in CPM:[1]

(DD) *Differential dynamics* (Stone 1989, pp. 124–125; Kellert 1993, pp. 50, 56–59).[2] There exists an algorithm which relates a state of the model at any given time to a state at any other time and the algorithm is not probabilistic.

(UE) *Unique evolution* (Stone 1989, pp. 124–125; Kellert 1993, pp. 50, 59–60). The model is such that a given state is always followed (preceded) by the same history of state transitions.

(VD) *Value determinateness* (Stone 1989, pp. 124–125; Kellert 1993, p. 50). Any state of the model can be described with arbitrarily small (nonzero) error.

(AP) *Absolute predictability* (Stone 1989, p. 128). Any state of the model can be generated from the algorithm with arbitrarily small (nonzero) error from any other state of the model.

Here a state of a model is taken to be characterized by the values of the positions and momenta of all the particles composing the model and corresponds to a point in phase space. The model is then taken to represent the physical systems of interest.

DD is motivated by actual physical theories expressed in terms of evolution equations. These equations are mathematical models for physical phenomena and, along with their initial and boundary conditions, are required to be nonprobabilistic expressing the Laplacean belief that there are no indeterministic elements in CPM like those present in collapse versions of quantum mechanics.

UE is closely associated with DD. It is the Laplacean belief that the models of CPM will repeat their behaviors exactly if the same initial conditions and boundary conditions are specified. For example the equations of motion for a frictionless pendulum will produce the same solution for the motion as long as the same initial velocity and initial position are chosen. Philosophers such as Jesse Hobbs (1991) and Kellert explicate this feature of determinism following John Earman:

> Letting W stand for the collection of all physically possible worlds, that is, possible worlds which satisfy the natural laws obtaining in the actual world, we can define the Laplacian [*sic.*] variety of determinism as follows. The world $w \in W$ is Laplacian [*sic.*] deterministic just in case for any $w' \in W$, if w and w' agree at any time, then they agree for all times. By assumption, the world-at-a-given-time is an invariantly meaningful notion and agreement of worlds at a time means agreement at that time on all relevant physical properties. (Earman 1986, p. 13)

Earman's formulation of Laplacean determinism is meant to be a metaphysical statement about our world. Briefly our world and any identical copies of it will have the same history. In order to avoid particular technical problems, I reformulate UE as follows (Bishop and Kronz 1999, pp. 130–132). Let M stand for the collection of all models sharing the same set L of physical laws and suppose that P is the set of relevant physical properties for specifying the time evolution of a model recognized by L. Then

> A model $m \in M$ exhibits *unique evolution* if and only if every model $m' \in M$ isomorphic to m with respect to P undergoes the same evolution as m (Bishop and Kronz 1999, p. 131).

The key idea for UE is that if we take a model to be the equations of motion, every time the model is returned to the same initial state, it will undergo the same history of state transitions. In other words the evolution of the model will be unique with respect to a particular specification of the initial and boundary conditions. Furthermore choose any state in the history of state transitions as the initial starting point and the model's history will remain unchanged forwards or backwards in time. The 'collection' of models is composed of all the isomorphisms allowed by the symmetry properties of the equations of motion. Hence exploring whether a given model has the property of unique evolution is the study of the equations of motion and their solution properties under the allowed symmetries.

The first two elements of the Laplacean vision correspond to a cluster of properties of mathematical equations known as well-posedness, continuity, and existence and uniqueness of solutions. These properties are required for unique solutions of the equations of mathematical physics to exist (Hadamard 1952; Birkoff and Rota 1978). The third element of this vision, VD, is motivated by the Laplacean belief that there is nothing *in principle* in CPM that prevents mathematical descriptions of arbitrary accuracy. For example the models of CPM all presuppose precise values for the constants and variables used in the equations of motion. Clark Glymour takes VD as one of the necessary criteria for determinism and cites Charles Peirce and Hans Reichenbach as examples of philosophers who have included this criterion in their analyses of determinism (1971, pp. 744–445). Note that Laplace in his famous (1951, p. 4) is explicitly committed to VD.

In addition mathematical models in CPM have sharp values as is implied by the equations of motion. The various uniqueness and existence theorems for these equations require definite initial values for such properties as position and momentum. Since CPM is often taken as the paradigm example of a deterministic theory, it is natural that value determinateness would come to be seen as part of the Laplacean vision for classical physics. It is only with the advent of quantum mechanics that scientists began raising questions about the applicability of VD to all of physics.

The final element of the Laplacean vision is AP. It is *prima facie* reasonable to expect that if the first three elements are in place, then at least *in principle* it should be possible to predict the behavior of any model and, in particular, the exact state a model would take on at any time (Laplace 1951, p. 4). AP should be construed as predictability *for all times* as the definition is independent of any references to specific times and makes no reference to other forms such as statistical predictability. In the Laplacean vision, DD, UE and VD are taken to imply AP. Furthermore, as I read Laplace, this is an *in principle* claim regardless of possible *in practice* limitations. Although Laplace believed that we could only approximate the capabilities of his hypothetical super-intelligence, he offers no evidence that our limitations are anything other than practical matters. Indeed his discussion of the progress of science in the introduction as well as in chapter 17 of *A Philosophical Essay on Probabilities* (1951) are consistent with the existence of *only* practical limitations to arbitrary accuracy in DD, UE and VD, and, hence, in our predictability. There are no in principle limitations to the methods of classical physics from Laplace's perspective.[3]

3. PREDICTABILITY AND DETERMINISM

Consider the following characterization of the concept of prediction capturing the intuitions found in DD, UE and VD. Let a measurement at time t be characterized by the ordered pair (φ^t, α), whose elements are a mapping φ^t and a fixed finite partition α (or carving up into phase cells) of the phase space Ω. The action of φ^t as t evolves carries a phase space point q into another phase space point q'. Let the model's state at t_0 be represented by a phase space point q_0. A prediction at t_i is a specification of which cell in α contains $\varphi_i^t(q)$. A series of predictions made at times t_i becomes a sequence of outcomes of the process (φ^t, α) for $t_0 < t < t_f$. The outcomes of these predictions correspond to a sequence of cell occupations characterized by giving probabilities for all finite strings of consecutive occupations. The prediction (φ^t, α) is *perfect* if the next occupation is fixed with probability 1 given at least some of the outcomes for past times. The mapping φ^t is perfect if, for all partitions α, (φ^t, α) yields a sequence of cell occupations each having probability 1. In this language AP corresponds to perfect predictability (φ^t, α), where φ^t can be related to DD, UE and VD, while α is related to error measurement/specification. In the Laplacean vision, that the phase cells can be made arbitrarily small is considered unproblematic.

Many attempts to dismiss any connection between determinism and predictability in nonquantum contexts turn out to be too quick. For example one might argue that determinism and predictability are separable, at least in some cases, due to "accidental reasons" (e.g., Bunge 1967, p. 81; and Bricmont 1995, p. 161). For example, it might be impossible to predict the time shown on a clock – a perfectly deterministic system whose model equations fulfill DD, UE and VD – if it is locked in a cabinet. However, this line of argument would only establish that prediction tasks may be *in practice* impossible to carry out in some contexts and would not rule out any *in principle* connections between DD, UE and VD on the one hand and AP on the other. Another way to argue for the separability of determinism and predictability is by noting that determinism is an ontological concept while predictability is an epistemic concept (e.g., Bunge 1967, p.

81; Earman 1986, pp. 7–8; Walter 2001/1999, pp. 17–20). In the Laplacean vision, however, prediction appears to be an ontological implication of DD, UE and VD even though it has epistemic import. Of course one could simply deny that some events which are determined are also predictable (e.g., Sobel 1998, p. 79), but this is precisely the question at stake.

A more substantial argument that AP is separable from the first three elements of the Laplacean vision, relying on chaotic dynamics, has been put forward by Stone (1989).[4] The key difficulty raised by Stone for AP amounts to the inability to define a small enough α (due either to inability to accurately sample or represent initial conditions) such that (φ^t, α) will yield cell occupations with fixed probability 1 for all times. He begins by taking the first three elements as each being necessary and together jointly sufficient conditions for a CPM model to be deterministic (here DD, UE and VD are properties of the group operators composing models in CPM; see Bishop 1999, pp. 30–32). That is to say, given a set of evolution equations and initial and boundary conditions, the trajectories and, hence, all states of the model are uniquely determined. According to Stone, however, determinism is only a necessary condition for AP. While the first three elements of the Laplacean vision are jointly sufficient for determinism, Stone argues they are not sufficient for AP.

His main argument proceeds along the following lines. The problem of prediction in classical physics boils down to the problem of overcoming the effects of error in $((\varphi^t, \alpha)$. Stone discusses two typical sources of inaccuracy and both types can introduce errors into predictions. The first source of error is due to limitations on the precision of measurements and the second is due to limitations on representing an initial value of a variable (e.g., velocity) to complete accuracy when that value is an irrational number such as $\sqrt{2}$ or π (both sources of error are discussed more fully in Appendix A). These errors can have four types of growth rates: (1) zero, where the error stays constant, (2) linear, (3) polynomial or (4) exponential. The last type is related to the presence of positive Lyapunov exponents (Hilborn 1994, pp. 138–140 and 171–178).

In the strongest form of predictability possible, there is an input accuracy ϵ such that for any output accuracy δ, our prediction accuracy δ_{pred} is always smaller than δ so that $\forall t \exists \epsilon \forall \delta (\delta_{\text{pred}}(\epsilon, t) < \delta)$. This form of predictability can be satisfied for error growth of types (2) – (4) *only if* $\epsilon = 0$. Clearly for the measurement and representation types of inaccuracies discussed in Appendix A, $\epsilon \neq 0$, so for any inaccuracies at all, this form of predictability can be satisfied *at best* only for zero error growth. Since the Laplacean vision considers the reduction of error to be an *in principle* kind of process, the more appropriate construal of AP – as critiqued by Stone – is of the form

$$(\text{AP}') \quad \forall\epsilon > 0\exists\delta\forall t\,(\delta_{\text{pred}}(\epsilon, t) < \delta).$$

Chaotic dynamics plays a crucial role in Stone's argumentation (1989, p. 127). For some parameter settings, chaotic dynamical models exhibit sensitivity to initial conditions with exponential error growth rates. Laplacean intuitions tell us that by making the initial error ϵ in our specification of the initial conditions precise enough, we can ensure that the error δ in the specification of any state is within the *in principle* limits of CPM as expressed by AP′. In contrast to this intuition, Stone argues that there is no procedure available within classical physics from which we can obtain $\delta_{\text{pred}}(\epsilon, t) < \delta$ from the initial specification of ϵ for chaotic models *for all times*.

He begins by supposing "we want to show that, with respect to a certain chaotic system, we can *always* get prediction limited to error measure [ϵ]" (1989, p. 127). Knowing that chaotic dynamics amplifies error at an exponential rate, the usual procedure would be to use classical physics to calculate the error in the specification of the future (past) state δ and then produce an ϵ such that $\delta_{\text{pred}}(\epsilon, t) < \delta$ is satisfied for some time t_1 say. The problem is that there is another time t_2 such that $\delta_{\text{pred}}(\epsilon, t_2) > \delta$ for the ϵ we chose. No matter how we choose ϵ, there will always be some t_i such that $\delta_{\text{pred}}(\epsilon, t_i) > \delta$ where $j > i$. In light of the limitations on accuracy discussed in Appendix A, no such ϵ exists guaranteeing $\delta_{\text{pred}}(\epsilon, t) < \delta$ for all t. Hence, any error in the specification of the initial conditions will be amplified by the equations of chaotic models such that AP′ cannot be fulfilled. It is assumed that chaotic models satisfy the first three conditions of the Laplacean vision and, hence, are still deterministic (an assumption called into question in Bishop and Kronz 1999). Chaotic models fail to satisfy AP′, therefore DD, UE, and VD taken together do not imply AP′. Determinism and absolute predictability are separable.

Although it is true that cases of chaos illustrate the separability of AP′ from determinism in a dramatic fashion, it turns out that Stone's reasoning applies to cases where the error grows linearly as well.[5] No ϵ can be found to fulfill AP′ in CPM models where even the slowest possible linear growth rate in the error occurs. The important difference is that *chaotic dynamics places much stricter limitations on the notion of predictability*. If the linear growth rate of a CPM model is small enough, perhaps we can calculate an ϵ that is valid for most times of interest to us except in cases where we ask what happens when t is allowed to grow very large. We cannot produce reliable predictions *for all times*, however, because even linear growth in initial errors eventually will exceed the accuracy bounds.

One form of criticism against Stone's line of argument found in the literature is the following. Even the most regular dynamical systems can exhibit complete unpredictability with respect to some set of variables, while, with respect to a (possibly) more relevant set of variables describing the system properties, predictability might be maintained (Batterman 1993, p. 45; Batterman and White 1996, pp. 307–308). The idea here is that arguments such as Stone's may turn on which set of variables are chosen. However, Stone's argument focuses on the model equations physicists use which presumably govern the appropriate system variables. Robert Batterman and Homer White make a suggestion that one might turn into a criticism of Stone's line of argument. Suppose that the true initial conditions in the system are x, yet y is our approximation of those conditions in a set $B_\epsilon(T, n, x)$,[6] then our prediction concerning the next n states of the system would differ from the truth by no more than ϵ (Brin and Katok 1983, pp. 30–38). Thus if we desire prediction accuracy to within ϵ, according to Batterman and White, all we need to do is "measure the initial conditions with sufficient accuracy so as to lie in $B_\epsilon(T, n, x)$" (1996, p. 318). The imagined objector might use this approach as an analysis for predictability in order to overcome Stone's argument. This suggestion, however, is dependent on the nature of T. If T amplifies error exponentially, then this approach, if extended for all times (all n), will fail for similar reasons as AP′ due to the in principle limitations on measurement and representation accuracy described in Appendix A.

It might be suggested that the following is a more appropriate construal of predictability for all times:

$$(\text{AP}^*) \quad \forall\epsilon > 0 \forall t \exists\delta(\delta_{\text{pred}}(\epsilon, t) < \delta).$$

Though AP* is weaker than AP′ (AP′ implies AP*) and is immune to Stone's line of argument, it clearly fails for chaotic contexts. In the case of Hamiltonian chaos, δ would have to be the size of the system (or in the best case the size of the relevant energy surface) in order to guarantee that $\delta_{\text{pred}}(\epsilon, t) < \delta$ is satisfied for all times rendering AP* trivial. In the case of dissipative chaos, δ would have to be the size of the attractor in order to guarantee that $\delta_{\text{pred}}(\epsilon, t) < \delta$ is satisfied for all times again rendering AP* trivial. Certainly AP* is implied by DD, UE and VD, but this is hardly what anyone had in mind as a construal of AP![7]

Does AP* fare any better for situations where the error growth rate is linear? Given enough time the error will be amplified, so that once again δ would have to be the size of the system (or in the best case the size of the invariant torus on which the trajectories are evolving) in order to guarantee that $\delta_{\text{pred}}(\epsilon, t) < \delta$ is satisfied for all times. More importantly it is possible to have strong mixing with zero Komolgorov entropy (Zaslavskiĭ and Chirikov 1972, p. 558; cf. Petersen 1999, pp. 62–63). Such systems would have error growth rates slower than exponential (i.e., polynomial) and would also require δ to be the size of the system (or in the best case the size of the relevant energy surface) in order to guarantee that $\delta_{\text{pred}}(\epsilon, t) < \delta$ is satisfied for all times. So it appears that AP* is trivial for nonexponential error growth as well.

A similar fate befalls a principle Smith calls

$$\text{(P)} \qquad \forall\delta\forall t\exists\epsilon(\delta_{\text{pred}}(\epsilon, t) < \delta).$$

As Smith points out, (P) implies that differential equations describing chaotic systems obey *continuous dependence on initial data* under rather mild assumptions (Smith 1998, pp. 54–55). If one wants to defend AP by using a version of Hadamard's condition (e.g., in the form of (P)), a possibility suggested by Schurz (1996, p. 135), or by a Lyapunov stability criterion (e.g., Earman 1986, pp. 162–163), the same fate as AP* awaits.

I believe the correct response to Stone's argument in the context of the Laplacean vision is to accept the separability thesis with respect to determinism and AP.[8] Classical physics does not guarantee arbitrary precision in the specification of future (past) states from the initial specification of ϵ for initial conditions except in a very trivial sense, so the Laplacean vision of in principle predictability for all times can only be fulfilled trivially at best.

Of course I have been discussing the simplest case of trajectories originating from nearby initial points in phase space. Typically chaotic systems have numerous bifurcation points as well (Hilborn 1994, pp. 118–126 and 599–604). Similar forms of argument as I have deployed regarding exponential divergence also apply at every bifurcation point in a system as even the smallest differences will determine which branch trajectories will take at a given bifurcation point.

Do DD, UE and VD imply any form of predictability? Under many circumstances it is possible to refine the specification of the initial conditions of some chaotic models so as to predict some future (past) states with arbitrary accuracy for some t such that

$$(\text{P}_t) \qquad \exists t\exists\delta\exists\epsilon(\delta_{\text{pred}}(\epsilon, t) < \delta).$$

These are finite prediction tasks and the time t is determined by the dynamics of the model involved. It may be too short for us to make useful predictions about events of interest to us in the future (past) of such models. For example integration of the equations of motion for the nine planets of the solar system indicate that the timescale for exponential divergence (and, hence, the predictability horizon before errors significantly affect accuracy) is approximately four million years (Sussman and Wisdom 1992). This means that predictions from such a calculation become unreliable on timescales very short with respect to the lifetime of the solar system due to limitations on measurement and representation accuracy of initial conditions.[9]

There is a further complication for predictability in the short run ignored by Stone's analysis. As is well known the exponential function is equivalent to an infinite series, i.e., $e^t = 1+t+t^2/2+t^3/6+\cdots t^n/n!+\cdots$. Furthermore it is easy to construct a polynomial in t that, for small values of t, actually has a much steeper growth rate than the exponential function (e.g., Schurz 1996, pp. 135–136). This means that although for large values of t exponential growth in errors will dominate, for small values of t steep polynomial growth in errors *may* dominate. Therefore in the short run, predictability problems can still be generated for models before the exponential growth due to chaos becomes a factor. Hence chaos is not needed to see that AP, under none of the above construals, can be a *bona fide* element of the Laplacean vision. Any error growth rate other than zero renders AP either unobtainable or trivial and can place significant constraints on P_t.

Nevertheless, the existence of a t for some error amplifying models, where future (past) states can be predicted within accuracy bounds, implies that some temporally indexed versions of predictability like P_t are *consistent* with the first three elements of the Laplacean vision. But constraints due to possible short term rapid polynomial amplification or long term exponential amplification show that even P_t is not implied by DD, UE and VD.

Given a set of model equations satisfying DD, UE and VD, an estimate of the measurement error and the representation constraints, we can calculate a time t_r up to which $\delta_{\text{pred}}(\epsilon, t_r) < \delta$. t_r, sometimes called the *relaxation time*, is a prediction horizon governed by the dynamics and error constraints, and can be calculated within the framework of CPM. Therefore classical physics does imply a limited formulation of prediction, namely

$$(\mathrm{P}_r) \qquad \forall t < t_r \exists\delta\exists\epsilon > 0(\delta_{\text{pred}}(\epsilon, t) < \delta).$$

P_r can be used along with DD, UE and VD to explicate a more limited version of the Laplacean vision. I believe this version more accurately represents the type of predictability as practiced in physics in the following sense. Rarely is it the case that predictions derived from theoretical results are treated as independent of all timescales as in the Laplacean vision; rather, they are typically tied to the timescales defined by the dynamics of the problem being solved or the timescales over which the physics is "interesting" (i.e., in the practice of physics, predictability is treated *ceteris paribus*). P_r also can make precise our intuitions that cases of exponential error amplification are of limited predictability while cases of nonexponential amplification are more robustly predictable. In addition I believe that P_r tracks with the kinds of breakdowns of CPM as discussed in Earman (1986), though the meaning of these breakdowns for determinism is a more delicate matter (Wilson 1989; Bishop 1999; Bishop and Kronz 1999).

Two curious facts about P_r relevant to the discussion of the relationship between determinism and predictability are worth pointing out. First, if a system or model is P_r-predictable, then it is deterministic. This is because t_r is a direct consequence of DD, UE and VD indicating that this form of predictability is ontological as well as epistemological in character. I should point out, however, that P_r also tells us its limits for deterministic systems, so it cannot be used à la Popper (1950) to argue that a failure of predictability implies a failure of determinism.[10] Second, though P_r is quite pragmatic, given that it is tied to our interests and abilities regarding model equations and irreducible errors, it is still rigorously derivable from those very model equations in conjunction with our best error estimates. The predictability horizon comes out naturally as a result of the calculations on the system in question. Time scales for when error amplification renders predictability practically useless would be much longer in nonexponential cases than in exponential cases, obviously, but the question of when the error amplification is too large to trust a given prediction is system dependent, purpose dependent and more of a practical matter. Therefore, predictability is not an "all or nothing" property of a system, but more a matter of degree.[11]

4. COMPELLING VISIONS DIE HARD

Although Stone's main argument seeking to demonstrate the separability of determinism and absolute predictability focused on chaotic dynamics, any error growth rate is sufficient to show that only a weaker form P_r-predictability can be consistent with determinism. Although clearly the Laplacean vision for determinism and predictability is too grandiose and must be tempered, those who dismiss any connection between determinism and predictability have also been too quick, particularly if they have some notion like AP in mind.

One response to this might be to say that it is obvious AP cannot be implied by DD, UE and VD. This is old news, so why dwell on the matter? In one respect such a response is right on the mark; namely, since models possessing error amplification have been around for a long time, AP should have been a deeply suspicious element of the Laplacean vision.

In another respect, however, this response is dead wrong. It is only *after* breaking free from the power of the full-orbed version of the Laplacean vision that one is able to see clearly that AP cannot be part of the vision *except* in a trivial sense. This vision has exerted a very strong influence on the practice of physics and other sciences as well as on philosophy up to the present day. One has only to examine standard physics textbooks still in use in university classrooms such as Symon (1971), Goldstein (1980) or Jackson (1975) to see that such influence continues in the training of physicists. Although many physicists do point out that chaotic dynamics presents problems for predictability, these problems are often taken to be only *in practice* limitations consistent with the Laplacean vision (e.g., Jensen 1987, p. 102; Ott 1993, pp. 17–18; Bricmont 1995, pp. 161–165). The search for deterministic explanations with absolute predictive power continues to plague the behavioral sciences (Slife and Williams 1995; Richardson and Bishop 2001). In philosophical discourse, Donald Davidson assumes some form of AP in his theory of events (Davidson 1980, p. 219). Jan Schmidt (1998a) has recently offered an argument purporting to solve Newcomb's paradox where a collection of minute observers are able to make exquisitely accurate measurements. Such an argument is fully under the spell of the Laplacean vision and, as such, is subject to the same restrictions as discussed in this essay.[12]

So the death of absolute predictability may be old news, but its message and meaning have yet to travel far. Of course one reason the message has not traveled far is that determinism is typically considered to be an ontological doctrine about the world. The truth of such an ontological doctrine could be taken as support for the idea that if the world is deterministic, then its systems should be predictable. I have not said anything about determinism as an ontological doctrine, but, rather, have confined my discussion to models and their corresponding systems. As my discussion demonstrates, however, even if this doctrine is true, P_r is the strongest form of prediction about the world's systems it could underwrite.

Although the kind of quantitative predictability implied by DD, UE and VD is highly constrained in cases of exponential amplification (even if metaphysical determinism is true), P_r is not useless in such cases.[13] Furthermore, by using the so-called shadow theorems (Smith 1998, pp. 58–60), pursuing statistical approaches or by studying the parameter values at which chaos appears in the model equations, useful qualitative information about the dynamics of chaotic systems can often be obtained. In addition P_r gives us a practical measure for when predictions can be usefully made. Properties such as being predictable or being unpredictable are time, system and purpose dependent rather than absolute as envisioned in the Laplacean Vision.

5. APPENDIX A: THE NATURE OF LIMITATIONS

There are two typical sources of inaccuracy and both types can introduce errors into predictions. The first source of error is due to limitations in the precision of measurements. In practice we do not measure the angular momentum of the moon as some value p but as $p + \epsilon$, where ϵ is the error in the accuracy of our measurement. The Laplacean vision holds that these limitations can be reduced so that *in principle* measurements can be made as precise as needed to achieve the desired prediction accuracy even if there are practical obstacles to achieving arbitrary accuracy. Stone states, "I am not convinced that in principle perfect measurement is possible" (1989, p. 125), but offers no argument as to why arbitrary measurement accuracy might be impossible to achieve. I will argue that there are fundamental limits on measurement accuracy and that these limits have consequences for predictability.

The usual assumption in CPM that measurement can be made arbitrarily accurate is not sufficient to overcome the effects of error growth because even the smallest amount of error in a measurement will be amplified. The argument that there are *in principle* limitations on measurement accuracy amounts to showing that it is impossible as a matter of principle to reduce measurement error to zero in CPM. The assumption that disturbances introduced by the act of measurement can be made arbitrarily small is also insufficient for the same reasons. If one wants to rule out the influence of error growth in predictability, one has to show that disturbances due to measurements in CPM can be reduced in principle to zero, a necessary – but not sufficient! – condition for perfect measurement accuracy. Measurement has to do with the error involved in determining the value of an initial state $\omega \in \Omega$ while predictability has to do with the propagation of that error under state evolution on the phase space Ω.

Suppose for the sake of argument that a measurement process can be devised that imparts no disturbance to a damped, driven pendulum exhibiting chaotic behavior. Is perfect measurement accuracy attainable? No. Although CPM assumes measurements can be made arbitrarily accurate, error reduction is a limiting process and at some point in the refinement of our measurement accuracy, the information storage needs will exceed the universe of discourse of CPM long before we reach infinite accuracy because the information needed to account for accuracy grows exponentially (quantum computing is of course ruled out in principle because it does not exist in the universe of discourse of CPM).[14]

Second chaotic systems amplify *any* perturbations no matter how small. For example under a conservative estimate an electron at the 'edge' of the known universe would affect a system of billiard balls undergoing continuous collisions, so even the minutest influences count (Crutchfield 1994, p. 239). Suppose we want to perform a perfect measurement on the position of a chaotic pendulum. The pendulum is continually bathed in perturbations due to continuous collisions with the air molecules in the chamber in which it is swinging; it is perturbed by the minute fluctuations in gravity due to the movements of the scientists and engineers carrying out the experiment; it is perturbed by the electromagnetic and gravitational interactions of cars, planes and trains passing near the building where the experiment is taking place; indeed it is perturbed by every other entity in the universe. The sensitive dependence of the chaotic pendulum will amplify all these perturbations.

So even if the disturbance of the measurement technique could be eliminated, the observers would still have to make a complete accounting for all the effects just mentioned (the position and motion of every body, every force, etc. in the universe) in order to make a perfect measurement of the chaotic pendulum's position for use in a prediction task. This is a large, though finite,[15] amount of information needed for a full accounting of an accurate measurement of the pendulum's position and certainly qualifies at least as *in practice* impossible.[16] If, as is no doubt the case, there are several other chaotic systems in the universe that must appear in our full accounting, then the information requirements become astronomical.

But that is "not the half of it" as the saying goes. According to James Crutchfield (1994, pp. 239–240), the observers (both humans and instruments) must also measure themselves for a full accounting of the exact state of the chaotic pendulum leading to an infinite regress of measurements measuring measurements requiring the storage of the information of the universe's state within a subsystem of it.

> The process of storage itself involves the measurement and manipulation of further subsystems' states. The infinite regression thus requires the storage of an infinite amount of information. Regardless of the size of the universe, this self-observation and internal self-coding is impossible. (1994, p. 240)

Given that there are multiple chaotic systems that must appear in our final accounting, the infinite regress of measurements and their infinite information requirements would be multiplied.

So exponential error growth limits in fundamental ways both the precision of measurements and predictability. Of course in the real world, we cannot ignore quantum mechanics. For example in measuring an aluminum bar, even if our measurement technique can be made as precise as we want, the quantum fluctuations of the bar will fundamentally limit how accurately we can measure its length. In the case of the chaotic pendulum, sensitive dependence implies that quantum effects can be amplified such that the motion of macroscopic objects like chaotic pendulums are affected (Bishop and Kronz 1999). Hence quantum level disturbances represent a further constraint on measurement accuracy in chaotic cases.

A second source of error is due to limitations on representing an initial value of a variable (e.g., velocity) to complete accuracy when that value is an irrational number. Stone takes this to be an *in principle* kind of limitation (without spelling out a supporting argument), but one might imagine such a limitation is merely due to computational resources (an *in practice* limitation). However, this is not the case. While it is true that computational resources may limit our ability to represent a string of digits of arbitrary length, the problem is that irrational numbers have an infinite string of digits. It is also true that there are some irrational numbers for which a representation in a Turing machine can be computed via an algorithm (so-called computable numbers), of which π is an example, but the vast majority of irrational numbers are noncomputable by any Turing machine (Pour-El and Richards 1989, pp. 13–24; Penrose 1989, pp. 49–87). So it is in principle impossible to represent an infinitely long string much less carry out calculations with such numbers. And *any* truncation of the string to finite length, no matter how long, will produce an error in the representation which will then be subject to growth as the dynamics unfold. Given that the set of rational numbers is far smaller than the set of irrational numbers, this representational limitation is more serious than often realized.

Smith has also given an argument purporting to demonstrate that arbitrary measurement accuracy is in principle impossible. He maintains that quantities like fluid velocity, temperature and chemical concentration do not possess indefinitely precise real number values in our best theories. Take fluid circulation velocity for example:

> We know ... that fluids are gappy distributions of molecules in motion. So, the circulation velocity of a fluid at a point P can be nothing other than the average velocity of all the molecules of the fluid contained in some small ball centered on P – or better, it might seem, the true circulation velocity is to be identified as the *limit* to which this average velocity tends as the size of the small ball around P shrinks to zero. But the trouble with the latter suggestion is, of course, that once we shrink the ball around P to the scale where it only contains a few dozen molecules, further shrinking will produce increasingly wild fluctuations due to the random Brownian motions of the molecules – and (in all probability) the final limit of the average velocity of the molecules in a very small ball around

P will be zero because there will be no molecules at all left in the ball. To avoid this absurdity in 'taking a limit', the best we could do is average the velocity of molecules in some arbitrarily selected standard volume (for example, a sphere of radius one micron centered at *P*). But even this still wouldn't produce perfect determinacy. For example, quantum indeterminacies on the usual understanding entail that there is no exact fact of the matter about e.g. which molecules have their centre of mass determinately in the ball at a particular time. In short, there seems no principled way completely precisifying the quantity *circulation-velocity-at-a-point*. (1998, pp. 39–40)

According to Smith, a similar story for the temperature a point *P* in a fluid can be spelled out, where the temperature is determined by the average kinetic energy of the molecules in some small ball centered on *P*. The upshot is that "*there is no fact of the matter* about the exact value of [such] quantities" (1998, p. 39).

However, this line of argument is both misdirected and misguided. Surely fluids have properties such as circulation velocity, temperature, chemical concentration and the like, and there is nothing about fluids *per se* to suggest that these properties lack exact values. Indeed, Smith's argument does not appear to be aimed at this point, but, rather, is more properly directed at the possibility of principled *epistemic* definitions of such quantities that would yield precise values. Understood in this way, Smith's considerations amount to no more than an in practice problem for making arbitrarily precise measurements of such quantities. But Smith's line of argument is also misguided. Algebraic statistical mechanics suggests that the type of molecular account Smith gives for quantities like circulation velocity, temperature and chemical concentration is not appropriate (e.g., Takesaki 1970; Müller-Herold 1980), calling the fundamental assumption of the argument into question.

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NOTES

[1] This is not to suggest that Laplace believed this vision could be carried out in practice with respect to absolute predictability. It is an *in principle* vision and has served more as a background assumption for scientific exploration whatever its practical qualifications.

[2] Both Stone and Kellert are ambiguous in their use of the term 'system' as to whether it denotes actual physical systems, the mathematical models for these systems or both. I will focus much of my attention on models.

[3] Reasons for *in principle* limitations on this vision are advanced in Appendix A below. N. G. van Kampen (1981) argued that the usual construal of Laplace's description of determinism was neither verifiable nor falsifiable and, hence, devoid of empirical content. As reformulated by Stone, Laplacean determinism is based directly on equations of motion for physical systems (with no omniscient observer assumed) and, therefore, is empirically rich in content. This form of determinism turns out to be false if AP is included.

[4] Richard Boyd (1972) pursued a different line of argument driving a wedge between determinism and AP based on showing that since there are only countably many finite sets of scientific laws, but a continuum of possible deterministic systems, not every deterministic system can correspond to a finite set of laws specifying its behavior exactly. I do not wish to pass judgment on this line of argumentation here. My interests are in showing that classical physics itself contains the necessary limitations on predictability (though these limitations have largely been ignored). Greg Hunt (1987) also discussed predictability and determinism in the context of chaos, but his essay has been sufficiently criticized in the literature (Batterman 1993, pp. 54–55; Smith 1998, pp. 56–58). In essence, Hunt's point that determinism and AP are separable is correct, but his arguments are faulty.

[5] As Zaslavskiĭ and Chirikov pointed out, any system undergoing nonlinear oscillations has an amplification rate proportional to t due to the frequency shifts between trajectories (1972, p. 558). Furthermore, many examples from the class of integrable systems have linear amplification rates. Or take Smith's simple example of a particle whose straight line motion is described by $x_0(1+t)$, which amplifies error linearly with time (1998, p. 58).

[6] Let X be a vector space equipped only with a metric d. Then define $B_\epsilon(T, n, x) = \{y \in X : d(T^j(x), T^j(y)) \leq \epsilon, 0 \leq j \leq n\}$, where T denotes a transformation, n the number of states and x, y represent actual and measured initial conditions of the system respectively.

[7] Olimpia Lombardi pointed out to me that this line of argument also applies equally against AP′.

[8] One should not interpret Batterman's discussion of Stone in (1993, pp. 52–54) as an argument showing that Stone's attempt to separate AP from the rest of the Laplacean vision fails. Batterman has in mind some more limited form of predictability, not AP′ which I think faithfully captures Stone's intended target.

[9] Thus Batterman's claim that "exponential instability is not really all that bad" in the case of finite prediction tasks because "it is, in fact, always possible to obtain *any* degree of accuracy in the specification of the final state by increasing the accuracy with which the initial state is specified" (1993, pp. 52–53, emphasis added) tends to underestimate the *in principle* limitations due to accuracy and representation errors discussed in Appendix A in the context of exponential instability.

[10] Russell appears to consider a form of prediction like AP, where determinism and predictability are equated (1953, p. 400), and suggests a possibility for overcoming one kind of in principle difficulty for such prediction. As Russell pointed out, defining a system as deterministic because it can be described by a function runs into difficulties (1953, pp. 401–402). At every instant in the history of the system, there exists an infinite number of functions describing the same time-evolution of the system in the past, but diverging in their descriptions in the future. Hence, this definition of determinism/prediction is vacuous because there is never a case where there is a single unique function describing the evolution. Russell's suggestion for overcoming this difficulty is to characterize systems by a function which does not explicitly refer to time. His hope was that such a move would diminish, if not totally remove, the number of possible functions that agree in their description of the time-evolution of the system up to the time t and disagree thereafter. To

my knowledge, this suggestion has never been worked out, but P_r calls the viability of such a move into question regarding predictability.

[11] There are additional questions regarding the computability of solutions to the evolution equations in mathematical models of physics (e.g., Earman 1986, pp. 111–127; Penrose 1989; Pour-EL and Richards 1989; Svozil 1993). In some examples computable initial data can be specified for the wave equation, but some solutions are *not* computable (Pour-El and Richards 1989, pp. 68–73 and 115–118). Although the initial data in these examples are not twice differentiable (typically physical fields are represented mathematically as being at least twice differentiable), such examples do raise questions as to whether under appropriate physical and mathematical assumptions, all model equations in physics have computable solutions. If computability failed for some of these models, it would then be the case that DD, UE and VD are satisfied while no forms of predictability are possible regardless of the errors made in measurement and representation.

[12] Schmidt basis his argument on a prediction theorem he proves and defends in Schmidt (1998b). Aside from a questionable assumption that particles in physically reasonable universes are collisionless (1998b, p. 84), his prediction theorem given on p. 87 of the latter essay assumes ϵ can go to zero in the limit (1998b, p. 91). Schmidt's unsubstantiated claim to the contrary, ϵ cannot be reduced to zero in CPM (Appendix A). It is not surprising that, in Schmidt's words, "Provided that the observational data is sufficiently accurate, arbitrary accurate predictions can be made" (1998b, p. 90), for this is the Laplacean intuition behind AP. Swept up in the full-orbed Laplacean vision, Schmidt downplays the severe limitations placed on predictability by observational and representational errors, so when his version of predictability is extended to indefinitely long intervals of time it becomes as trivial as AP*.

[13] Kellert's use of such phrases as "predictive hopelessness" (1993, p. 33) and "utterly unpredictable" (1993, p. 62) to characterize predictability in chaotic systems should be understood in the context of some form of AP.

[14] Kellert (1993, pp. 29–42) makes a similar argument with more details. Though there may be some question in his presentation about how to distinguish various forms of impossibility, we both agree that perfect measurement is more than *in practice* impossible.

[15] Assuming a finite universe. If the universe is assumed to be infinite (as is often the context in carrying out calculations in classical electrodynamics and Newtonian gravity) then an infinite amount of information is needed for a full accounting and the *in principle* nature of the limitation on perfect measurement is obvious.

[16] It is actually *in principle* impossible as a full accounting for the state of the universe requires an observer outside the universe to make the needed measurement and CPM offers no prescription for how such a measurement can in principle be carried out.

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