Functional limits of zeta type processes

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Abstract

The Riemann zeta process is a stochastic process $\{Z(\sigma),\, \sigma > 1\}$ with independent increments and marginal distributions whose characteristic functions are proportional to the Riemann zeta function along vertical lines $\Re s = \sigma$. We establish functional limit theorems for the zeta process and other related processes as arguments σ approach the pole at $s = 1$ of the zeta function (from above).

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1 Introduction

The Riemann zeta function is defined in the half-plane $\Re s = \sigma > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$ (product over all primes). Along vertical lines $\Re s = \sigma > 1$, the renormalized zeta function $\zeta_\sigma(t) = \zeta(\sigma + it)/\zeta(\sigma)$, $t \in \mathbf{R}$, represents the characteristic function of a discrete probability distribution putting weights $n^{-\sigma}/\zeta(\sigma)$ at points $-\log n$ $(n \geq 1)$, as is clear from the first expression for the zeta function. By the second expression, this distribution is infinitely divisible, an observation going back to Khintchine (1938). See [11] and [9, p. 75], respectively.

Given this family of "zeta distributions" indexed by the parameter $\sigma > 1$, it is natural to ask for a stochastic process with time parameter σ having these distributions as its marginals. Such a "zeta process" was constructed by Alexander, Baclawski and Rota [1] for discrete time indices $\sigma = 2, 3, \ldots$. Recently we proposed an elementary construction of a *continuous time* zeta process [6], thus making it possible to "zoom in" on $\sigma = 1$ and study the zeta process locally at this singularity of the zeta function. This could be of great interest particularly if it would suggest a way to continue the zeta process to the region $\sigma \leq 1$. Here we establish some functional limit theorems including, in particular, a non-Gaussian, "scaled exponentials" limit for the zeta process and functional versions of Erdős-Kac type theorems. Chance enters these results via the stochastic interpretation of the zeta function, while in the density theorems of number theory it emerges "empirically" through relative frequencies.

Various number-theoretical and stochastic aspects of the (one-dimensional) zeta distribution, including its representation as a linear combination of independent geometric random variables and related generalizations, are discussed in [11] and [10]. A survey of statistical interpretations of the zeta function and some of its relatives emphasizing surprising connections with Brownian motion may be found in [2]. Systematic accounts of probabilistic number theory are given in [7] and [14].

The paper is organized as follows. In Section 2 we recall the definition and basic properties of the zeta process and of the geometric process it is derived from. The functional limit theorems for the zeta process and other related processes appear in Section 3. The final Section 4 provides an alternative proof for the zeta limit law and discusses the rôle of the prime number theorem in that context.

2 The geometric process, and the zeta process

In [6], we constructed a *geometric process* $Y = \{Y(u),\ u \in [0,1)\}$ specified by the following properties (G1), (G2).

(G1) Y has non-decreasing, right-continuous sample paths starting at $Y(0) = 0$.

(G2) Y is a Markov process with state space $\mathbf{Z}_+$ and transition probabilities given, for $0 \le u < w < 1$ and $m = 0, 1, \ldots,$ by

$$P[Y(w) = m + n \,|\, Y(u) = m] \;=\; \begin{cases} \frac{1-w}{1-u} & \text{if} \quad n = 0 \\ \frac{w-u}{1-u}\,(1-w)\,w^{n-1} & \text{if} \quad n \ge 1\,. \end{cases}$$

The transition probabilities are independent of the respective state m. Therefore, Y is a process with *independent increments.* The increments are not homogeneous in "time", however: $Y(w) - Y(u)$ equals zero with probability $(1-w)/(1-u)$ and is positive with the complementary probability $(w-u)/(1-u)$, according to the geometric distribution with parameter w shifted by $+1$. Mean and variance of the increments are given by

$$E\,(Y(w) - Y(u)) \;=\; \frac{w-u}{(1-u)(1-w)} \tag{1}$$

$$\mathrm{Var}\,(Y(w) - Y(u)) \;=\; \frac{(w-u)(1-uw)}{(1-u)^2(1-w)^2}\,, \tag{2}$$

respectively, for $0 \le u \le w < 1$. Marginally, $Y(u)$ is geometrically distributed with parameter u, $P[Y(u) = n] = (1-u)\,u^n$ $(u \in [0,1);\, n \in \mathbf{Z}_+)$. Its characteristic function thus is $E\,\mathrm{e}^{itY(u)} = (1-u)/(1-u\,\mathrm{e}^{it})$, $t \in \mathbf{R}$.

The *(Riemann) zeta process* $Z = \{Z(\sigma),\ \sigma > 1\}$ is linearly assembled from geometric processes each of which undergoes a (different) time change. Toward the definition of Z let $\{Y_p\}$ be a family of independent copies of the geometric process indexed by the primes. Then *with probability 1 it holds that for every $\sigma > 1$ there are at most finitely many primes p such that* $Y_p(p^{-\sigma}) \neq 0$. Indeed, let us consider for every fixed $\tau > 1$ the event $A_\tau = [\,\sup_{\sigma \ge \tau} Y_p(p^{-\sigma}) > 0 \ f.i.m.p]$ where f.i.m.p means "for infinitely many p". The monotonicity property (G1) implies that A_τ is identical with the event $B_\tau = [\,Y_p(p^{-\tau}) > 0 \ f.i.m.p]$. But $P(B_\tau) = 0$ by the Borel-Cantelli Lemma, since $\sum_p P\,[Y_p(p^{-\tau}) > 0] = \sum_p p^{-\tau} < \infty$ $(\tau > 1)$. Therefore, $P(A_\tau) = 0$ for every $\tau > 1$, and hence $P\,[\cup_{n \ge 1} A_{1+n^{-1}}] = 0$, proving the above claim.

Given independent geometric processes Y_p we may now define the zeta process by setting

$$Z(\sigma) = \sum\nolimits_p Y_p(p^{-\sigma})\,\log p\,, \quad \sigma > 1. \tag{3}$$

The preceding consideration shows that almost surely the sum over all primes involves only finitely many nonzero members for every $\sigma > 1$, so that the trajectories $\sigma \rightsquigarrow Z(\sigma)$ are well-defined almost surely. Note that (3) may also be stated in the form $Z(\sigma) = \log N(\sigma)$

where $N(\sigma) = \prod_p p^{Y_p(p^{-\sigma})}$. Therefore, by the uniqueness of the prime factorization the component processes $\sigma \rightsquigarrow Y_p(p^{-\sigma})$ can be reconstructed from the zeta process.

The representation of $Z(\sigma)$ as a linear combination of independent geometric random variables with parameters $p^{-\sigma}$ implies that its characteristic function splits into the product $E\mathrm{e}^{itZ(\sigma)} = \prod_p (1-p^{-\sigma})/(1-p^{-\sigma}\,\mathrm{e}^{it\log p})$, which equals the renormalized zeta function up to a sign change, $E\mathrm{e}^{itZ(\sigma)} = \zeta_\sigma(-t)$. The expectation and variance of $Z(\sigma)$ are ([11]), respectively,

$$E\,Z(\sigma) \;=\; \sum\nolimits_p p^{-\sigma}\,(1-p^{-\sigma})^{-1}\,\log p \;=\; -(\log\zeta)'(\sigma)\,, \tag{4}$$

$$\mathrm{Var}\,(Z(\sigma)) \;=\; \sum\nolimits_p p^{-\sigma}\,(1-p^{-\sigma})^{-2}\,\log^2 p \;=\; (\log\zeta)''(\sigma)\,. \tag{5}$$

The new feature compared to [11], [10] is that the random variables $Z(\sigma), \sigma > 1$, all live on a common probability space, with the arguments of the Y_ps now representing (transformed) “time” in addition to their rôle as parameter of a geometric distribution.

The basic properties of the zeta process readily follow from those of the geometric process. In particular, like the latter, the zeta process has independent increments, or better, since transformed “times” $p^{-\sigma}$ run backwards, *independent decrements.* The basic properties of the zeta process are essentially retained if the coefficients $\log p$ are replaced by general coefficients c_p, as we will do subsequently.

3 Functional limit theorems

Let (c_p) be a sequence of real constants indexed by the primes. Given (c_p), let the functions $\mu(\sigma)$, $V(\sigma)$ be defined by

$$\mu(\sigma) \;=\; \sum\nolimits_p c_p\, p^{-\sigma}, \qquad V(\sigma) \;=\; \sum\nolimits_p c_p^2\, p^{-\sigma} \qquad (\sigma > 1).$$

The definition makes sense under the first of the following conditions (C), which are assumed to hold throughout the sequel.

(C) $\quad V(\sigma) < \infty\ \forall\,\sigma > 1$ (i); $\quad \lim_{\sigma\downarrow 1} V(\sigma) = \infty$ (ii); $\quad \lim_{\sigma\downarrow 1} \frac{\sup_q c_q^2\, q^{-\sigma}}{V(\sigma)} = 0$ (iii).

Let Y_p $(p = 2, 3, 5, \ldots)$ be independent geometric processes. The process

$$(1,\infty) \ni \sigma \rightsquigarrow M(\sigma) \;=\; \sum\nolimits_p c_p\,\big(Y_p(p^{-\sigma}) - p^{-\sigma}/(1-p^{-\sigma})\big)$$

is well-defined for the same reason as in the case of the zeta process. It is centered and has independent decrements, hence is a backwards martingale with respect to the natural filtration $\mathcal{G} = \{\mathcal{G}(\sigma), \sigma > 1\}$ where $\mathcal{G}(\sigma) = \bigvee_p \mathcal{F}_p(p^{-\sigma})$, with $\mathcal{F}_p(w) = \sigma\{Y_p(u), 0 \le u \le w\}$, $0 \le w < 1$. Its variance function is

$$\mathrm{Var}\,M(\sigma) \;=\; \sum\nolimits_p c_p^2\, p^{-\sigma}/(1-p^{-\sigma})^2.$$

To formulate the basic functional limit theorem it is convenient to reverse the “time” direction once again, to “forward”, by passing to time parameters $u \in [0,1]$ as follows. The function $\sigma \mapsto V(\sigma)$ decreases strictly and continuously on $(1,\infty)$ from $V(1+) = \infty$ to $V(\infty) = 0$, by condition (C). Therefore, given $\tau > 1$ there exists for every $u \in (0,1]$

a uniquely defined $\sigma_\tau(u) \geq \tau$ such that $V(\sigma_\tau(u)) = u\,V(\tau)$. Then $\sigma_\tau(u)$ decreases from ∞ to $\sigma_\tau(1) = \tau$ as u runs from 0 to 1 (and conversely), and we may set $\sigma_\tau(0) = \infty$.

For any $\tau > 1$, let the process ξ_τ be defined by

$$\xi_\tau(u) \;=\; V(\tau)^{-1/2}\, M(\sigma_\tau(u)) \qquad (0 \leq u \leq 1)$$

where we set $\xi_\tau(0) = 0$ in accordance with $V(\infty) = 0$ and $E\,\xi_\tau(u)^2 = u \downarrow 0$ as $u \downarrow 0$. Weak convergence of processes is understood in the space $D[0,1]$ of right-continuous functions with left-hand limits endowed with the Skorokhod topology.

Theorem 3.1 *Let the constants c_p satisfy condition (C). Suppose there exists a finite, nonnegative measure with distribution function K such that*

$$\lim\nolimits_{\sigma\downarrow 1} K_\sigma(b) \;=\; K(b) \tag{6}$$

for every continuity point b of K, where

$$K_\sigma(b) \;=\; V(\sigma)^{-1} \sum\nolimits_p c_p^2\, p^{-\sigma}\, \chi(c_p \leq b\, V(\sigma)^{1/2}) \qquad (\sigma > 1,\, b \in \mathbf{R})$$

and $\chi(\cdot)$ denotes the indicator function of the event in brackets. Then as $\tau \downarrow 1$ the process $\xi_\tau = \{\xi_\tau(u),\, 0 \leq u \leq 1\}$ converges weakly to the process $\xi = \{\xi(u),\, 0 \leq u \leq 1\}$ characterized by the following properties: (i) ξ has independent increments; (ii) for $u \in [0,1]$ the distribution of $\xi(u)$ is infinitely divisible with log characteristic function $\lambda_u(t) = \lambda(t\sqrt{u}\,)$, where

$$\lambda(t) \;=\; \int_0^\infty (\mathrm{e}^{itx} - 1 - itx)\, x^{-2}\, dK(x) \qquad (t \in \mathbf{R}). \tag{7}$$

The theorem represents a strictly stochastic, functional relative of classical density theorems in number theory developed by Turán, Erdős, Kac, Kubilius, and others; cf. Elliott [7, Ch. 12]. Those results deal with the asymptotic value distribution of arithmetical functions $f(n)$ that are additive in the sense that $f(n) = \sum_p f(p)\, y_p$ or $f(n) = \sum_p f(p)\chi(y_p \geq 1)$ if $n = \prod_p p^{\,y_p}$ is the prime factorization of n. Distribution here refers to the relative frequency of numbers n in the range $1, \ldots, N$ having the property under consideration, and the asymptotics is for $N \to \infty$. Functional limit theorems in this setting, with Brownian motion as the limit process, were obtained by Philipp [13], who points to related earlier work of Billingsley, Kubilius, and others. These results refer to the actual frequency distribution of the prime numbers, hence are "harder" than ours where numbers are sampled randomly and statistical independence is built in from the beginning. In turn, our approach is very simple technically and more directly related to the zeta function.

The proof of Theorem 3.1 relies on a well-known criterion ([9, p. 100, Theorem 2]) for convergence in law to an infinitely divisible distribution with finite variance. We state it in a way adapted to Brown and Eagleson's [5] generalization of the result to martingale difference arrays.

Theorem 3.2 ([9], [5]) *For each $n \in \mathbf{N}$ let $X_{n,k}$, $k = 1, 2, \ldots$ be a (possibly infinite) sequence of independent random variables satisfying $EX_{n,k} = 0$, $\sum_k EX_{n,k}^2 \leq C < \infty$, and $\max_k EX_{n,k}^2 \to 0$ as $n \to \infty$. Suppose there exists a finite measure with distribution function G such that for every continuity point b of G*

$$\lim_{n\to\infty} \sum\nolimits_k E\, X_{n,k}^2\, \chi(X_{n,k} \leq b) \;=\; G(b).$$

Then as $n \to \infty$, $\sum_k X_{n,k}$ converges in law to the infinitely divisible distribution with log characteristic function given by $\int (\mathrm{e}^{itx} - 1 - itx)\, x^{-2}\, dG(x)$.

A second ingredient is an approximation using clipped variables $\widetilde{Y}_p(u) = Y_p(u) \wedge 1 = \min\{Y_p(u), 1\}$. This modification of the $Y_p(u)$s is inessential for the limit theorems due to the following.

Lemma 3.3 *Let $S(\sigma) = \sum_p c_p\, Y_p(p^{-\sigma})$, $\widetilde{S}(\sigma) = \sum_p c_p\, \widetilde{Y}_p(p^{-\sigma})$.*
If the constants c_p satisfy conditions (C) then uniformly in $\sigma > 1$

$$E\, \widetilde{S}(\sigma) \;=\; \mu(\sigma) \;=\; E\, S(\sigma) + O(1), \tag{8}$$

$$\operatorname{Var} S(\sigma) \;=\; V(\sigma) + O(1), \quad \operatorname{Var} \widetilde{S}(\sigma) \;=\; V(\sigma) + O(1). \tag{9}$$

Moreover,

$$E\, \sup\nolimits_{\sigma > 1} |S(\sigma) - \widetilde{S}(\sigma)| \;<\; \infty. \tag{10}$$

Proof. The difference $Y_p(u) - \widetilde{Y}_p(u)$ is non-negative and non-decreasing in u, whence

$$|S(\sigma) - \widetilde{S}(\sigma)| \;\le\; \sum\nolimits_p |c_p| \left(Y_p(p^{-\tau}) - \widetilde{Y}_p(p^{-\tau}) \right)$$

for $\sigma \ge \tau > 1$. Clearly $E\, \widetilde{Y}_p(u) = u$, hence $E\, \widetilde{S}(\sigma) = \mu(\sigma)$. Since $p^{-\sigma} \le 1/2$ for all primes and $\sigma \ge 1$, and $0 \le E\, Y_p(u) - E\, \widetilde{Y}_p(u) = u/(1-u) - u \le 2\,u^2$ for $0 \le u \le 1/2$, it follows by the monotonicity of V and Cauchy-Schwarz that

$$\begin{aligned} E\, \sup\nolimits_{\sigma \ge \tau} |S(\sigma) - \widetilde{S}(\sigma)| \;&\le\; \sum\nolimits_p |c_p|\, E \left(Y_p(p^{-\tau}) - \widetilde{Y}_p(p^{-\tau}) \right) \;\le\; 2 \sum |c_p|\, p^{-2\tau} \\ &\le\; 2 \left(V(2) \sum p^{-2} \right)^{1/2} \;<\; \infty. \end{aligned}$$

The upper bound being independent of $\tau > 1$, we obtain (10) by taking the limit $\tau \downarrow 1$ on the left hand side and using monotone convergence. This also proves (8), and (9) follows easily on observing that $\operatorname{Var} Y_p(u) = u/(1-u)^2$ and $\operatorname{Var} \widetilde{Y}_p(u) = u(1-u)$.

A useful consequence of the lemma and condition (Cii) is the asymptotic equivalence

$$\sup\nolimits_{0 \le u \le 1} |\widetilde{\xi}_\tau(u) - \xi_\tau(u)| \;=\; o_p(1) \quad \text{as} \quad \tau \downarrow 1 \tag{11}$$

of the processes ξ_τ and the "clipped processes" $\widetilde{\xi}_\tau$ defined by

$$\widetilde{\xi}_\tau(u) \;=\; V(\tau)^{-1/2} \left(\widetilde{S}(\sigma_\tau(u)) - \mu(\sigma_\tau(u)) \right) \qquad (0 \le u \le 1).$$

Thus any result (regarding weak convergence) that holds for one of the two processes also holds for the other one.

Proof of Theorem 3.1. Tightness of ξ_τ follows from a fluctuation inequality of Billingsley; cf. [3, Theorem 15.6] and the subsequent remark. Indeed, by the independence of the increments of ξ_τ we have for $0 \le u_1 \le u \le u_2 \le 1$

$$\begin{aligned} & E\, [(\xi_\tau(u) - \xi_\tau(u_1))^2\, (\xi_\tau(u_2) - \xi_\tau(u))^2] \\ =\; & V(\tau)^{-2}\, (V(\sigma_\tau(u)) - V(\sigma_\tau(u_1)))\, (V(\sigma_\tau(u_2)) - V(\sigma_\tau(u))) \\ =\; & (u - u_1)\, (u_2 - u) \;\le\; (u_2 - u_1)^2, \end{aligned}$$

as required.

Again by the independence of the increments, for convergence of the finite-dimensional distributions it suffices to show that $\xi_\tau(u)$ or, equivalently, $\widetilde{\xi}_\tau(u)$ converges in law to a variable with log characteristic function λ_u, for each $u \in [0,1]$. The case $u = 0$ being clear, let us fix $u \in (0,1]$ and set

$$H_v(b,\sigma) \;=\; v^{-1} \sum\nolimits_p c_p^2 \, E\left[X_p(\sigma)^2 \, \chi\Big(c_p \, X_p(\sigma) \le b v^{1/2}\Big)\right] \qquad (v > 0,\ \sigma > 1,\ b \in \mathbf{R})$$

where $X_p(\sigma) = \widetilde{Y}_p(p^{-\sigma}) - p^{-\sigma}$. Suppose we can show that

$$\lim\nolimits_{\tau\downarrow 1} H_{V(\tau)}(b, \sigma_\tau(u)) \;=\; u\, K(bu^{-1/2}) \;=:\; K_u(b) \tag{12}$$

for every continuity point b of K_u. Then the conditions of Theorem 3.2 are satisfied with $X_{n,k} \equiv c_p \, X_p(\sigma_\tau(u))/\sqrt{V(\tau)}$ and $G \equiv K_u$, and we may conclude that $\sum_k X_{n,k} \equiv \widetilde{\xi}_\tau(u)$ converges in law to the distribution with log characteristic function

$$\int (\mathrm{e}^{itx} - 1 - itx)\, x^{-2}\, u \, dK(xu^{-1/2}) \;=\; \int (\mathrm{e}^{it\sqrt{u}\, y} - 1 - it\sqrt{u}\, y)\, y^{-2}\, dK(y) \;=\; \lambda(t\sqrt{u}),$$

as desired. The second reduction step consists in showing that if suffices to prove (12) for $u = 1$; i.e., since $\sigma_\tau(1) = \tau$, to prove that for every continuity point b of K we have

$$\lim\nolimits_{\tau\downarrow 1} H_{V(\tau)}(b, \tau) \;=\; K(b). \tag{13}$$

Indeed, from $V(\sigma_\tau(u)) = uV(\tau)$ we at first get the scaling relation

$$\begin{aligned}
& H_{V(\tau)}(b, \sigma_\tau(u)) \\
=\; & V(\tau)^{-1} \sum\nolimits_p c_p^2 \; E\,[X_p(\sigma_\tau(u))^2 \; \chi(c_p \, X_p(\sigma_\tau(u)) \le b\, V(\tau)^{1/2})] \\
=\; & \frac{u}{V(\sigma_\tau(u))} \sum\nolimits_p c_p^2 \; E\left[X_p(\sigma_\tau(u))^2 \; \chi\bigg(c_p \, X_p(\sigma_\tau(u)) \le \frac{b}{u^{1/2}}\, V(\sigma_\tau(u))^{1/2}\bigg)\right] \\
=\; & u \; H_{V(\sigma_\tau(u))}\Big(b\, u^{-1/2}, \sigma_\tau(u)\Big)\;.
\end{aligned}$$

Secondly, $V(\sigma_\tau(u))$ tends to infinity along with $V(\tau)$ as $\tau \downarrow 1$, which by condition (Ci) is possible only if $\sigma_\tau(u) \downarrow 1$. Therefore (13) implies that for every continuity point b of K_u, or continuity point $bu^{-1/2}$ of K,

$$\lim\nolimits_{\tau\downarrow 1} H_{V(\tau)}(b, \sigma_\tau(u)) \;=\; \lim\nolimits_{\tau\downarrow 1} u\, H_{V(\sigma_\tau(u))}\Big(b\, u^{-1/2}, \sigma_\tau(u)\Big) \;=\; u\, K(bu^{-1/2}).$$

Thus the case $u \in (0,1)$ is reduced to the case $u = 1$, and it remains to prove (13).

To that end let us evaluate the expectations of the Bernoulli variables in $H_{V(\tau)}(b,\tau)$. Writing $X_p = X_p(\tau)$, $V = V(\tau)$ for ease of notation, we have

$$\begin{aligned}
\sum\nolimits_p c_p^2 \, E\, X_p^2 \, \chi(c_p X_p \le b\sqrt{V}) \;&=\; \sum\nolimits_p c_p^2 \, (1 - p^{-\tau})\, p^{-2\tau} \, \chi(-c_p \, p^{-\tau} \le b\sqrt{V}) \\
&\qquad + \sum\nolimits_p c_p^2 \, p^{-\tau} \, (1 - p^{-\tau})^2 \, \chi(c_p \, (1 - p^{-\tau}) \le b\sqrt{V}) \\
&=\; A_0(\tau, b) \;\;+\;\; A_1(\tau, b),
\end{aligned}$$

and thus

$$H_{V(\tau)}(b,\tau) \;=\; V^{-1}A_0(\tau,b) \,+\, V^{-1}A_1(\tau,b).$$

The first term is negligible since $A_0(\tau,b) \le \sum_p c_p^2\, p^{-2\tau} = O(1) = o(V)$. As for the second term, let $\widetilde{p} = \widetilde{p}_\tau$ tend to infinity so slowly that $\sum_{p\le\widetilde{p}} c_p^2\, p^{-\tau} = o(V)$ as $\tau \downarrow 1$. Then given any $\epsilon > 0$, one has $p^{-\tau} < \widetilde{p}^{-1} < \epsilon$ for all $p > \widetilde{p}$ and $\widetilde{p}$ large enough, hence

$$\begin{aligned}
V^{-1} A_1(\tau,b) &\le V^{-1}\sum\nolimits_{p>\widetilde{p}} c_p^2\, p^{-\tau}\,\chi(c_p \le (1-\epsilon)^{-1}\, b\sqrt{V}) \,+\, V^{-1}\sum\nolimits_{p\le\widetilde{p}} c_p^2\, p^{-\tau}\\
&= K_\tau(b/(1-\epsilon)) \,+\, o(1),\\
V^{-1} A_1(\tau,b) &\ge (1-\epsilon)^2\, V^{-1}\sum\nolimits_{p>\widetilde{p}} c_p^2\, p^{-\tau}\,\chi(c_p \le b\sqrt{V})\\
&= (1-\epsilon)^2 K_\tau(b) \,-\, o(1).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, these bounds together with (6) imply that (13) holds at every continuity point b of K. The proof of Theorem 3.1 is complete.

Let us now turn to two special cases of particular interest. To apply Theorem 3.1 to the zeta process we set $c_p = \log p$ and consider the process $u \rightsquigarrow \eta_\tau(u)$ defined by

$$\eta_\tau(u) \;=\; \sqrt{u} \,+\, V(\tau)^{-1/2}\, R(\sigma_\tau(u)), \qquad \text{where} \qquad R(\sigma) \;=\; Z(\sigma) + (\log\zeta)'(\sigma)$$

is the centered zeta process; cp. (4). The drift term $\sqrt{u}$ is added in order to undo the in this case unnecessary centering: it follows from (16) below, (5), and (9) that $-(\log\zeta)'(\sigma) = (\sigma-1)^{-1} + O(1) = V(\sigma)^{1/2} + O(1)$, and hence by the definition of $\sigma_\tau(u)$, that $V(\tau)^{-1/2}\,(\log\zeta)'(\sigma_\tau(u)) = -\sqrt{u} + o(1)$. Consequently,

$$\eta_\tau(u) \;=\; V(\tau)^{-1/2}\, Z(\sigma_\tau(u)) + o_p(1) \;=\; (\tau-1)\, Z(\sigma_\tau(u)) + o_p(1) \tag{14}$$

uniformly in u. Therefore *the process* $u \rightsquigarrow \eta^\star_\tau(u) = (\tau-1)\, Z(\sigma_\tau(u))$ *has the same weak limit as the process* η_τ.

The following lemma is standard, however, a concise reference is not easily found. The respective first estimates follow on observing (cf. [15, p. 46]) that the difference $\log\zeta(s) - \sum_p p^{-s}$ represents an analytic function in the region $\Re s > 1/2$ (and taking derivatives at $\sigma > 1$). The respective second estimates follow similarly on noting that by the elementary analytic properties of the zeta function ([14, p. 139, Theorem 1]) the function $h(s) = (s-1)\zeta(s)$ is analytic throughout the complex plane with $h(1) = 1$, then passing to logarithms (and their derivatives).

Lemma 3.4 *As* $\sigma \downarrow 1$ *one has*

$$\log\zeta(\sigma) \;=\; \sum\nolimits_p p^{-\sigma} \,+\, O(1) \;=\; -\log(\sigma-1) \,+\, O(1); \tag{15}$$

$$(-1)^k\,(\log\zeta)^{(k)}(\sigma) \;=\; \sum\nolimits_p p^{-\sigma}\log^k p \,+\, O(1) \;=\; \frac{(k-1)!}{(\sigma-1)^k} \,+\, O(1) \quad (k\ge 1). \tag{16}$$

Theorem 3.5 *As* $\tau \downarrow 1$ *the process* $\eta_\tau = \{\eta_\tau(u),\, 0\le u\le 1\}$ *converges weakly to the process* $\eta = \{\eta(u),\, 0\le u\le 1\}$ *characterized by the following properties: (i)* η *has independent increments; (ii) for* $u \in [0,1]$, $\eta(u)$ *is exponentially distributed with expectation* $\sqrt{u}$, $\eta(u) \sim \mathcal{E}(\sqrt{u})$.

Proof. Conditions (C) are satisfied for $c_p = \log p$. Let $V = V(\tau)$, and let $\alpha = \alpha_\tau$ tend to zero so slowly that $\alpha\sqrt{V} \to \infty$, yet fast enough that $\sum_{\log p \le \alpha\sqrt{V}} p^{-\tau} \log^2 p = o(V)$ as $\tau \downarrow 1$. Finally, let

$$\widetilde{K}_\tau(b) \;=\; V^{-1} \sum\nolimits_p p^{-\tau} \log^2 p \;\; \chi(\alpha\sqrt{V} < \log p \le b\sqrt{V}) \qquad (\tau > 1,\; b \in \mathbf{R}).$$

By Theorem 3.1 and the choice of α it suffices to show that

$$\lim\nolimits_{\tau\downarrow 1} \widetilde{K}_\tau(b) \;=\; (1-(1+b)\,\mathrm{e}^{-b})_+ \;=:\; K(b) \tag{17}$$

for every $b \in \mathbf{R}$ and to note, observing (7) and $K'(x) = x\,\mathrm{e}^{-x}$, that

$$\int_0^\infty (\mathrm{e}^{it\sqrt{u}\,x} - 1 - it\sqrt{u}\,x)\,\mathrm{e}^{-x}\,\frac{dx}{x} \;=\; \log\,\frac{\mathrm{e}^{-it\sqrt{u}}}{1-it\sqrt{u}}$$

is the log characteristic function of an $\mathcal{E}(\sqrt{u})$–distributed random variable minus $\sqrt{u}$. For the proof of (17) we may assume $b > 0$ because for $b \le 0$ both $\widetilde{K}_\tau(b)$ and $K(b)$ vanish. Denoting by $\pi(x)$ the prime counting function we may write $\widetilde{K}_\tau(b)$ as a Stieltjes integral and apply partial integration (actually: partial summation; slight inaccuracies with boundary terms do not matter and will be ignored). Thus we get

$$\begin{aligned}\widetilde{K}_\tau(b) \;&=\; V^{-1} \int_{\mathrm{e}^{\alpha\sqrt{V}}}^{\mathrm{e}^{b\sqrt{V}}} x^{-\tau} \log^2 x \; d\pi(x) &(18)\\ &=\; V^{-1} \left(x^{-(\tau-1)}\,\rho(x)\,\log x \,\Big|_{\mathrm{e}^{\alpha\sqrt{V}}}^{\mathrm{e}^{b\sqrt{V}}} \right) \;+\; V^{-1} \int_{\mathrm{e}^{\alpha\sqrt{V}}}^{\mathrm{e}^{b\sqrt{V}}} \rho(x)\,x^{-\tau}\,(\tau\log x - 2)\,dx\,,\end{aligned}$$

on writing $\rho(x) = x^{-1}\,\pi(x)\log x$ in the last line. The prime number theorem implies $\rho(x) = 1 + o(1)$ uniformly within the integration range. It follows that the boundary term is $O(V^{-1/2}) = o(1)$ and that the integral term equals $1 + o(1)$ times

$$V^{-1} \int_{\mathrm{e}^{\alpha\sqrt{V}}}^{\mathrm{e}^{b\sqrt{V}}} x^{-\tau}\,(\tau\log x - 2)\,dx \;=\; \int_\alpha^b \mathrm{e}^{-s(\tau-1)\sqrt{V}}\,(\tau s - 2V^{-1/2})\,ds\,,$$

after an obvious variable substitution. By (16), $V = (\log\zeta)''(\tau) = (\tau-1)^{-2} + O(1)$, hence $(\tau-1)\sqrt{V} = 1 + o(1)$; moreover, $\alpha = o(1)$. Therefore,

$$\lim\nolimits_{\tau\downarrow 1} \int_\alpha^b \mathrm{e}^{-s(\tau-1)\sqrt{V}}\,(\tau s - 2V^{-1/2})\,ds \;=\; \int_0^b \mathrm{e}^{-s} s\,ds \;=\; K(b).$$

This proves (17) and, hence, the theorem.

Remark 3.6 The limit process η reflects the structure of the geometric process. In fact, independence of the increments of η implies that for $0 \le u < w \le 1$ we have $E\,\exp[it(\eta(w)-\eta(u))] = (1-it\sqrt{u})/(1-it\sqrt{w})$, which quotient is readily identified as the characteristic function of the mixture distribution $\sqrt{u/w}\,\delta_0 + (1-\sqrt{u/w})\,\mathcal{E}(\sqrt{w})$. An analogous mixture representation by the point mass at zero and a geometric distribution was found for the increments of the geometric process.

A different type of (Gaussian) limits obtains if we set $c_p = 1$ for all p. Here $S(\sigma) = \sum_p Y_p(p^{-\sigma})$ represents the number of prime factors of $N(\sigma) = \prod_p p^{Y_p(p^{-\sigma})}$ (counting multiplicities), and the ensuing result may be regarded as a strictly stochastic, functional version of the celebrated Erdős-Kac theorem [8], [13]. Given $\tau \in (1,2)$ and $u \in [0,1]$ let

$$\beta_\tau(u) \;=\; V(\tau)^{-1/2}\, M(\sigma_\tau(u)) \quad \text{where} \quad M(\sigma) \;=\; S(\sigma) - E\,S(\sigma).$$

Note that similarly as at (14)

$$\beta_\tau(u) \;=\; \frac{S(\sigma_\tau(u)) \,+\, u\log(\tau-1)}{\sqrt{-\log(\tau-1)}} \;+\; o_p(1) \tag{19}$$

uniformly in u as $\tau \downarrow 1$ since by (15) one has $V(\tau) = -\log(\tau - 1) + O(1)$, and hence

$$E\,S(\sigma_\tau(u)) \;=\; V(\sigma_\tau(u)) + O(1) \;=\; uV(\tau) + O(1) \;=\; -u\log(\tau-1) + O(1). \tag{20}$$

Thus the concrete representation (19) may be substituted in the statement of the following Erdős-Kac type theorem.

Theorem 3.7 *As $\tau \downarrow 1$ the process $\beta_\tau = \{\beta_\tau(u),\, 0 \le u \le 1\}$ converges weakly to standard Brownian motion.*

Proof. Conditions (C) are satisfied. For $b \neq 0$ the limit

$$K(b) \;=\; \lim\nolimits_{\sigma\downarrow 1}\, V(\sigma)^{-1} \sum\nolimits_p \, p^{-\sigma}\chi(1 \le b\sqrt{V(\sigma)})$$

exists and equals 0 for $b < 0$ and 1 for $b > 0$. Thus K is the distribution function of δ_0, the unit point mass at zero, and (7) evaluates to $\lambda(t) = -t^2/2$, implying a standard normal limit law. The result then follows from Theorem 3.1.

The analogy with the Erdős-Kac theorem is enhanced if the summation is restricted to primes $p \le q$ where $q \to \infty$. Sums being finite we may go down to $\sigma = 1$, and we set $S_q(\sigma) = \sum_{p\le q} Y_p(p^{-\sigma})$, $\sigma \ge 1$. The further procedure is analogous to the one followed so far. Given q and $u \in [0,1]$ we define $\sigma_q(u) \ge 1$ by the relation $V_q(\sigma_q(u)) = uV_q(1)$ where $V_q(\sigma) = \sum_{p\le q} p^{-\sigma}$. The expression $V_q(1)$ can be evaluated by means of a classical result of Mertens (cf. [12] or [14, p. 16]) which implies

$$V_q(1) \;=\; \sum\nolimits_{p\le q} p^{-1} \;=\; \log\log q \,+\, O(1) \quad \text{as} \quad q \to \infty.$$

Since both the expectation and variance of $S_q(\sigma)$ equal $V_q(\sigma) + O(1)$, arguing as at (20) we get $E\,S_q(\sigma_q(u)) = u\log\log q \,+\, O(1)$, so that

$$\omega_q(u) \;:=\; \frac{S_q(\sigma_q(u)) - E\,S_q(\sigma_q(u))}{\sqrt{V_q(1)}} \;=\; \frac{S_q(\sigma_q(u)) \,-\, u\log\log q}{\sqrt{\log\log q}} \;+\; o_p(1) \tag{21}$$

uniformly in u. This concretizes the following version of an Erdős-Kac type theorem.

Theorem 3.8 *As $q \to \infty$ the process $\omega_q = \{\omega_q(u),\, 0 \le u \le 1\}$ converges weakly to standard Brownian motion.*

One may wish to try the above reasoning also with the similarly truncated zeta process. However, there are difficulties due to its substantially different asymptotic behavior depending on whether q grows "slowly" or "fast". Roughly, if we consider the truncated process in the range $1 \le \sigma \le \tau$ and q grows slowly in the sense that $(\tau-1)\log q = o(1)$ as $\tau \downarrow 1$ and $q \to \infty$, then the limit is Gaussian; see Proposition 3.9 below. On the other hand, if q grows fast, the restriction $p \le q$ becomes inessential and the limit behavior is described by Theorem 3.5. Consequently, the behavior in the intermediate range is subtle.

The case of a slow growth presents no difficulties. Let $Z_q(\sigma) = \sum_{p\le q} Y_p(p^{-\sigma}) \log p$, $\sigma \ge 1$, denote the truncated zeta process.

Proposition 3.9 *Let* $\tau \downarrow 1$ *and* $q \to \infty$ *such that* $(\tau-1)\log q \to 0$. *Then*

$$\sup_{1\le\sigma\le\tau} |Z_q(\sigma) - Z_q(1)| \;=\; o_p(\log q) \tag{22}$$

and

$$\frac{Z_q(1) - \log q}{\log q} \;\longrightarrow_d\; \mathcal{N}(0, 1/2). \tag{23}$$

Proof. Since $Z_q(\sigma)$ is decreasing in σ, the supremum in (22) equals $Z_q(1) - Z_q(\tau)$. By (1) there is a finite constant C such that

$$\begin{aligned} E\,(Z_q(1) - Z_q(\tau)) \;&\le\; C\sum\nolimits_{p\le q} \log p\;(p^{-1} - p^{-\tau}) \;=\; C\sum\nolimits_{p\le q} \log^2 p \int_1^\tau p^{-\sigma}\,d\sigma \\ &\le\; C\,(\tau-1)\sum\nolimits_{p\le q} p^{-1}\log^2 p \;=\; O((\tau-1)\log^2 q) \;=\; o(\log q). \end{aligned}$$

For the last two steps we used (24) below and the "slow growth" assumption. By (2) we similarly get

$$\mathrm{Var}\,(Z_q(1) - Z_q(\tau)) \;\le\; C\sum\nolimits_{p\le q} \log^3 p \int_1^\tau p^{-\sigma}\,d\sigma \;=\; O((\tau-1)\log^3 q) \;=\; o(\log^2 q),$$

which together with the first estimate proves (22). The approximations to the expectation and variance of $Z_q(1)$ required for (23) follow from the asymptotic relations

$$\sum\nolimits_{p\le q} p^{-1}\log p \;=\; \log q + O(1), \quad \sum\nolimits_{p\le q} p^{-1}\log^2 p \;=\; (1+o(1))\,\tfrac{1}{2}\log^2 q \tag{24}$$

as $q \to \infty$. The first one is due to Mertens, cf. [12] or [14, p. 14]. The second is less sharp; it can be proved similarly as at (18) by writing the sum as an integral w.r.t. the prime counting function and integrating by parts. An application of the central limt theorem completes the proof.

Remark 3.10 As a consequence of Lemma 3.3 and the divergence of the scaling constants V, all results of this section remain valid if variables $Y_p(u)$ throughout are replaced by variables $\widetilde{Y}_p(u) = Y_p(u) \wedge 1 = \min\{Y_p(u), 1\}$. In particular, our versions of the Erdős-Kac theorem hold in entirely analogous form if the number of prime factors is counted *disregarding* multiplicities.

4 An alternative proof

The proof of Theorem 3.5 relies on the prime number theorem. By contrast, Billingsley's proof of the Erdős-Kac theorem uses the method of moments. The latter in fact applies also with the non-Gaussian limit of the zeta process.

Second Proof of Theorem 3.5. It suffices to show that $(\sigma-1)Z(\sigma)$ converges to the standard exponential distribution as $\sigma \downarrow 1$; cp. (14). Let $\widetilde{Z}(\sigma) = \sum_p \epsilon_p\, c_p$ where $c_p = \log p$ and $\epsilon_p = \widetilde{Y}_p(p^{-\sigma})$ are independent Bernoulli variables with success probabilities $E\epsilon_p = p^{-\sigma}$. Then $Z(\sigma) = \widetilde{Z}(\sigma) + O_p(1)$ for $\sigma \downarrow 1$ by Lemma 3.3, which reduces the task to showing that for every $m = 1, 2, \ldots$ one has

$$\lim_{\sigma\downarrow 1}\ E\,\Big((\sigma-1)\,\widetilde{Z}(\sigma)\Big)^m \ =\ \int_0^\infty x^m\, \mathrm{e}^{-x}\, dx \ =\ m!\,. \tag{25}$$

Following Billingsley [4, p. 391] we write

$$\widetilde{Z}(\sigma)^m \ =\ \sum_{k=1}^m \sum{}' \frac{m!}{r_1!\cdots r_k!}\,\frac{1}{k!} \sum{}'' (\epsilon_{p_1} c_{p_1})^{r_1} \cdots (\epsilon_{p_k} c_{p_k})^{r_k} \tag{26}$$

where $\sum'$ extends over the k-tuples $(r_1\, \ldots, r_k)$ of integers $r_j \geq 1$ satisfying $r_1 + \ldots + r_k = m$ and $\sum''$ extends over the k-tuples $(p_1\, \ldots, p_k)$ of pairwise distinct primes p_j. The expectation of the last sum in (26) equals

$$\sum{}'' p_1^{-\sigma} c_{p_1}^{r_1} \cdots p_k^{-\sigma} c_{p_k}^{r_k} \ =\ \sum{}^{(1)} \prod_{j=1}^k p_j^{-\sigma} c_{p_j}^{r_j} \ -\ \sum{}^{(2)} \prod_{j=1}^k p_j^{-\sigma} c_{p_j}^{r_j} \tag{27}$$

where $\sum^{(1)}$ denotes the unrestricted sum over all k-tuples $(p_1\, \ldots, p_k)$ of primes, and $\sum^{(2)}$ the sum over all such k-tuples for which at least two entries p_i, p_j are identical. Using (16) and substituting $c_p = \log p$ one finds that the first sum in (27) equals

$$\begin{aligned}\prod_{j=1}^k \Big(\sum_p p^{-\sigma}\, \log^{r_j} p\Big) \ &=\ \prod_{j=1}^k \left(\frac{(r_j-1)!}{(\sigma-1)^{r_j}} + O(1)\right)\\ &=\ (\sigma-1)^{-m} \prod_{j=1}^k (r_j-1)! \ +\ O((\sigma-1)^{-m+1}).\end{aligned}$$

The second sum is bounded by a product of $< k$ factors of the form $\sum_p p^{-\nu\sigma}\, \log^r p$ for some $1 \leq \nu,\, r \leq m$, with at least one of those factors having $\nu \geq 2$. Any such term with $\nu \geq 2$ stays bounded as $\sigma \downarrow 1$, so the second sum is at most of the order $O((\sigma-1)^{-m+1})$ in this limit. Consequently, putting everything together we obtain

$$\lim_{\sigma\downarrow 1}\ E\,\Big((\sigma-1)\,\widetilde{Z}(\sigma)\Big)^m \ =\ \sum_{k=1}^m \sum{}' \frac{m!}{r_1!\cdots r_k!}\, \frac{(r_1-1)!\cdots(r_k-1)!}{k!}\,. \tag{28}$$

It remains to verify that the last expression simplifies to $m!$. This, however, follows by decomposing the permutations of the set $\overline{m} = \{1, \ldots, m\}$ into products of disjoint cycles. For every $k = 1, \ldots, m$ there are $m!/(r_1!\cdots r_k!)$ choices of k nonempty, disjoint subsets of $\overline{m}$ consisting of $r_1, \ldots, r_k$ elements, respectively. Any such subset gives rise to $(r_j-1)!$ cycles that can be freely combined to give $(r_1-1)!\cdots(r_k-1)!/k!$ permutations. The division by $k!$ accounts for the irrelevance of the order in which the k cycles are composed. Thus the right-hand side of (28) counts the number of permutations of the set $\overline{m}$, which is $m!$. The proof is complete.

Having seen that the limit theorem for the zeta process does not depend on the prime number theorem [PNT], one may ask whether conversely, the PNT might follow from the limit theorem. A possible clue is furnished by a known theorem concerning weak limits of infinitely divisible distributions. It says, roughly, that convergence of such distributions is equivalent to convergence of the associated Lévy-Kolmogorov measures; cf. [9, p. 91, Theorem 3]. To apply that theorem we rely on the following alternative represention of the zeta process, which may be of interest on its own.

Proposition 4.1 *For primes p and $n = 1, 2, \ldots$ let $\xi_{p,n}$ be independent Poisson processes with respective intensities n^{-1}. Then*

$$(1, \infty) \ni \sigma \ \rightsquigarrow \ Z_\infty(\sigma) \ = \ \sum_p \sum_{n \geq 1} n \log p \ \xi_{p,n}(p^{-\sigma n})$$

represents a Riemann zeta process. The truncated version $Z_1(\sigma) = \sum_p \log p \ \xi_{p,1}(p^{-\sigma})$ approximates $Z_\infty(\sigma)$ in the sense that $Z_\infty(\sigma) - Z_1(\sigma) = O_p(1)$ uniformly in $\sigma > 1$.

Proof. In view of (3) the first assertion is an immediate consequence of the following alternative representation of the geometric process. *If $\xi_n = \{\xi_n(u), u \geq 0\}$ are independent Poisson processes with respective intensities n^{-1}, $n \geq 1$, then the process Y defined by*

$$Y(u) \ = \ \sum_{n \geq 1} n \, \xi_n(u^n), \qquad 0 \leq u < 1, \tag{29}$$

represents a geometric process. Indeed, Y is by construction a process with independent increments starting at $Y(0) = 0$. Therefore, it suffices to note that for any $u \in [0, 1)$ the characteristic function of $Y(u)$,

$$\begin{aligned} E \, \mathrm{e}^{itY(u)} \ &= \ \prod_n E \, \mathrm{e}^{it \, n \, \xi_n(u^n)} \ = \ \exp\Big[\sum_n \frac{u^n}{n} (\mathrm{e}^{itn} - 1)\Big] \\ &= \ \exp\left[- \log(1 - u \, \mathrm{e}^{it}) + \log(1 - u)\right] \ = \ \frac{1 - u}{1 - u \, \mathrm{e}^{it}} \ , \end{aligned}$$

equals the characteristic function of a geometric random variable with parameter u.
For the second assertion, note that $Z_\infty(\sigma) - Z_1(\sigma)$ is nonnegative and decreasing as a function of $\sigma > 1$, and that

$$E \left(Z_\infty(\sigma) - Z_1(\sigma) \right) \ = \ \sum_p \log p \sum_{n \geq 2} n \, \frac{p^{-\sigma n}}{n} \ \leq \ 2 \sum_p p^{-2} \log p$$

for every $\sigma > 1$.

A first consequence of Proposition 4.1 is the limit law

$$(\sigma - 1) \, Z_1(\sigma) \ \longrightarrow_d \ \mathcal{E}(1) \quad \text{as} \quad \sigma \downarrow 1. \tag{30}$$

Secondly, the process Z_1 has compound Poisson, hence infinitely divisible marginal distributions. The logarithm of the rescaled characteristc function equals

$$\log E \, \mathrm{e}^{it(\sigma-1) Z_1(\sigma)} \ = \ \sum_p p^{-\sigma} \left(\mathrm{e}^{it(\sigma-1)\log p} - 1 \right) \ = \ \int_0^\infty (\mathrm{e}^{itx} - 1 - itx) \, x^{-2} dF_\sigma(x) + it\gamma_\sigma$$

where $\gamma_\sigma = (\sigma - 1) \sum_p p^{-\sigma} \log p$ and F_σ denotes the Lévy-Kolmogorov measure given by the discrete mixture

$$F_\sigma \; = \; (\sigma-1)^2 \sum\nolimits_p p^{-\sigma} \log^2 p \; \delta_{(\sigma-1)\log p} \, .$$

Given these facts the theorem cited above, [9, p. 91, Theorem 3], implies that as $\sigma \downarrow 1$ quantities γ_σ and F_σ converge to the corresponding quantities associated with the limit distribution $\mathcal{E}(1)$. That is, $\gamma_\sigma \to 1$ (this also would follow from (16), of course); and in particular,

$$F_\sigma(b) \; = \; (\sigma-1)^2 \sum\nolimits_{\log p \le b/(\sigma-1)} p^{-\sigma} \log^2 p \;\to\; 1-(1+b)\mathrm{e}^{-b} \qquad (b \ge 0). \tag{31}$$

Going back to the proof of Theorem 3.5 we see that this last convergence is essentially (17), derived here without recourse to the PNT, however. (Just identify σ and τ, and note that then $V^{-1} \sim (\sigma-1)^2$; the initial segment corresponding to $\log p \le \alpha/(\sigma-1)$ is negligible.) A partial integration as at (18) and omission of the asymptotically negligible terms show that (31) implies, again with $\rho(x) \; = \; x^{-1}\pi(x)\log x$,

$$\lim\nolimits_{\sigma\downarrow 1} \;\; (\sigma-1)^2 \int_2^{\mathrm{e}^{b/(\sigma-1)}} \rho(x)\, x^{-\sigma} \; \log x \, dx \;\; = \;\; 1-(1+b)\mathrm{e}^{-b}. \tag{32}$$

(The PNT cannot be used here to prove that the boundary term, e.g., is asymptotically negligible. However, Chebyshev's classical estimates $c_1(1+o(1)) \le \rho(x) \le c_2(1+o(1))$ as $x \to \infty$ [14, p. 10] are amply sufficient for that purpose.)

Does the PNT follow from (32)? Substituting $x = \mathrm{e}^{t/(\sigma-1)}$ in the integral in (32) one finds that (32), hence (31), amount to

$$\lim\nolimits_{\sigma\downarrow 1} \int_{(\sigma-1)\log 2}^{b} \rho\Big(\mathrm{e}^{t/(\sigma-1)}\Big) \; \mathrm{e}^{-t}\, t\, dt \;\; = \;\; \int_0^b \mathrm{e}^{-t}\, t\, dt \, . \tag{33}$$

The conclusion is, not surprisingly, that the limit law (30) does not imply the PNT: although (33) shows that $\rho(x)$ approaches 1 in an average sense as $x \to \infty$, the "speeding up" of the argument allows for oscillatory behavior of ρ incompatible with the PNT.

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