Options for Testing Temporal Bell Inequalities for Mental Systems

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Abstract. It is shown how the concept of Bell inequalities may be used to decide whether "superposition" states exist in mental systems. For this purpose a generalized form of temporal Bell inequalities, originally developed for two-state systems, is derived for systems with any finite number of states. We propose options for testing violations of these inequalities in psychological experiments and discuss the important role of "non-invasive" measurements. Classical models can violate temporal Bell inequalitites, but observations are invasive.

Keywords: entanglement, invasiveness, neural networks, non-commutativity, temporal Bell inequalities, temporal nonlocality

1 Introduction

From an algebraic viewpoint, the main difference between the mathematical formalism of classical physics and quantum physics is the non-commutativity of observables. In the usual framework of classical physics, observables are functions on phase space (or, more general, configuration space) with commutative pointwise product. The essential physical reason for this commutative behavior is the fundamental assumption of classical physics that observations have no influence on the state of an observed system, in particular they do not change this state.

In the mathematical formulation of quantum theory, measurements are represented by (linear, self-adjoint) operators acting on the space of states usually assumed to be a Hilbert space (i.e., essentially a vector space with a scalar product defined for vectors). This representation of observables takes into account the experimental evidence that observations (or measurements) change the state of a system. Therefore, the results of and, particularly, the resulting state after temporally successive measurements may depend on the order of the measurements. This is reflected by the non-commutativity of the mathematical representations of observables.

In the mathematical framework of quantum theory, the non-commutativity of observables is related to all other non-classical phenomena of quantum systems, such as intrinsic indeterminism, superposition states, quantum probabilities, uncertainty relations, and the violation of so-called Bell inequalities. However, non-commuting observables alone do not strictly imply these quantum features. In quantum theory, the set of observables fulfills many more conditions, and the axiomatic definition of states as expectation value functionals for observables leads to an almost fixed mathematical structure.

Recently, there have been attempts to generalize the mathematical framework of quantum theory (see [1, 2]), and it is still an open question under which conditions the typical features of non-classical behavior are to be expected.

In mental systems it is obvious that observations generically influence the observed system by changing its state. Therefore, it is to be expected that observations do not commute and that a mathematical representation of these observables has to involve non-commutative structures. However, the extent to which such a non-commutative structure of mental observables leads to non-classical behavior remains open.

The "holy grail" for evident non-classical behavior would be a violation of Bell inequalities [3]. They refer to correlations between the results of measurements of different observables, and they have to be satisfied by any system for which the result of any measurement is strictly determined by the present state of the system. The assumption of such a strict determination alone is sufficient to derive Bell inequalities, not only for physical systems but for any system for which notions such as state, observable, measurement, and so on, make sense – including mental systems. A violation of Bell inequalities in mental systems would yield far-reaching insights into the nature of mental states.

In Sec. 2 we will derive a class of Bell inequalities which is particularly suited for temporal correlations between observations. Section 3 emphasizes the "non-invasiveness" of the measurements necessary to observe a possible violation of Bell-type inequalities experimentally. We will briefly describe classical models for which a non-commutative structure for observables can be defined but where a violation of Bell-type inequalities is merely the result of invasive observations. A brief summary and outlook conclude the article.

2 Bell Inequalities

In this section we will derive Bell inequalities which are particularly suited to be tested in experiments where the different observables correspond to the same type of measurements. We will first derive a conventional Bell inequality for a simple two-state system in subsection 2.1. Then, in subsection 2.2, we consider temporal Bell inequalitites for such a system, and in subsection 2.3 we discuss a more general temporal Bell inequality for an arbitrary (discrete) number of states, in particular for cases in which not all possible states are known.

2.1 Bell Inequality for a Two-State System

We assume that three observables are given with respect to which a system can be in one of two possible states characterized by + and −.[1] Later, in subsections 2.2 and 2.3, we will consider temporal versions of Bell inequalities where the measurements refer to the same experiment, but are performed successively at different moments in time.

The central assumption will be that the state of a system determines the outcome of each of the three measurements. This implies that each possible state belongs to one of eight classes, each class being labeled by the possible outcomes of the measurements (see left hand side of Table 1). It should be emphasized that we do not assume that in a particular situation we actually *know* to which class the momentary state of a system belongs. In particular, for mental systems it will be almost impossible to determine the class to which the mental state of an individual belongs. For the following arguments, it is sufficient that such an assignment of a state to one of the classes is possible in principle.

s_1	s_2	s_3	$N^-(1,3)$	$N^-(1,2)$	$N^-(2,3)$
+1	+1	+1			
+1	+1	−1	×		×
+1	−1	+1		×	×
+1	−1	−1	×	×	
−1	+1	+1	×	×	
−1	+1	−1		×	×
−1	−1	+1	×		×
−1	−1	−1			

Table 1. Any classical system falls into one of eight possible classes with respect to the three measurement results s_i, $i = 1, 2, 3$. Crosses under $N^-(i,j)$ mark those cases where the results s_i and s_j of measurements differ.

Table 1 shows that in all cases for which measurement 1 and measurement 3 yield *different* results, either measurement 1 and 2 yield different results, $s_1 \neq s_2$, or measurement 2 and 3 yield different results, $s_2 \neq s_3$. Moreover, there are also cases for which $s_1 \neq s_2$ *or* $s_2 \neq s_3$, but $s_1 = s_3$.[2] We can now deduce from Table 1 that for any given ensemble of states the following inequality holds,

$$N^-(1,3) \leq N^-(1,2) + N^-(2,3)\,, \tag{1}$$

where $N^-(i,j)$ denotes the total number of states for which measurement i and measurement j ($i, j = 1, 2, 3$ and $i \neq j$) yield different results.

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[1] The result of a single measurement for each of the observables can also be "yes" or "no", or "0" or "1", or, more generally, a and b.

[2] The "or" here is the logical OR, not the logical "exclusive or" XOR.

Inequality (1) is already a Bell inequality. In principle, this inequality can be tested by determining, for a given ensemble of systems (e.g., a group of individuals), the numbers $N^-(i,j)$ and then checking the results. If we assume that the probability for being in one of the states does not depend on the particular members of the ensemble but is a general property of mental systems (with inter-individual variability), we can interpret inequality (1) as a probability relation for a particular population:

$$p^-(1,3) \leq p^-(1,2) + p^-(2,3)\,. \tag{2}$$

This means, for each individual we only measure whether or not two of the measurements (with randomized sequence) yield different results, but we do not have to determine separately to which of the eight classes the (mental) state of each individual belongs.

It can be shown that a violation of Bell inequalities is only possible, if the three measurements do not commute. In this case quantum theory tells us that it is not possible for a state to determine the outcome with respect to each of the three measurements. (Technically speaking, there are no simultaneous eigenstates for observables which do not commute.) Therefore, the initial assumption (each state belonging to one of the eight possible classes) must be wrong. This makes the non-commutativity of observables a necessary prerequisite for a possible violation of Bell inequalities. For quantum systems this has been empirically confirmed beyond any reasonable doubt [4].

The non-commutativity of quantum observables makes it impossible to experimentally determine the precise outcomes of the corresponding measurements simultaneously. For our argumentation, however, it was important that a measurement does not change the state of a system in such a way that a second measurement yields a result different from the one it would have yielded in case the first measurement had not been performed.[3] This condition is called the "non-invasiveness" of a measurement.

In order to have non-invasive measurements (at least in a classical meaning), Bell proposed to test inequality (2) for entangled states, where only one measurement has to be performed on each subsystem. Entanglement guarantees a correlation allowing us to deduce the state of one subsystem from the measured state of the other subsystem. If the subsystems are sufficiently separated and the measurements of the two subsystems are performed almost simultaneously, the assumption of non-invasiveness is classically justified. In quantum theory, however, each of the measurements leads to a (non-local) change of the total state. Therefore, on a quantum level these measurements are invasive.

For mental systems we have no evidence for entangled states (e.g., between different individuals). Therefore, non-invasive measurements cannot be guaranteed this way. We will come back to this point in more detail in subsection 2.3.

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[3] Even if all changes were deterministic, the class to which a state belongs depends on the order in which the measurements are performed.

2.2 Temporal Bell Inequalitites for Two-State Systems

Since non-commuting observables are a prerequisite for a violation of Bell's inequalities, good candidates for such observables need to be selected. Instead of trying to measure two of three non-commuting observables at a time (and having to circumvene the uncertainty relations), a more suitable test for violations of Bell inequalities in mental systems is realized by so-called temporal Bell inequalities.[4] The actual measurements remain the same as discussed above, but the different observables now are measured at three different instances in time. If the temporal evolution of a system is incompatible with an observable (i.e., if the Hamiltonian does not commute with that observable), the observables corresponding to measurements at different instances may not commute. For such a situation we can reformulate the Bell inequality (1) in the following form:

$$N^-(t_1,t_3) \leq N^-(t_1,t_2) + N^-(t_2,t_3)\,, \tag{3}$$

where now $N^-(t_i,t_j)$ refers to the number of cases where a measurement at t_i and a second measurement at t_j yield different results. The essential assumption — which generalizes the assumption that a state determines uniquely the outcome of any measurement — now is that the history of a system is fixed and that the history of the system determines the outcome of any measurement at any time.

If we assume time-translation invariance and choose the instances in such a way that $t_3 - t_1 = 2(t_2 - t_1) = 2\tau$, we obtain [7]:

$$N^-(2\tau) \leq 2N^-(\tau)\,. \tag{4}$$

This is a sublinearity condition saying that the number of cases for which different results have been obtained with a time interval 2τ should always be smaller (or at least equal) twice the number of cases for which different results have been obtained with a time interval τ. Inequality (4) is the one we will discuss in the context of possible experiments with mental systems in order to test violations of Bell inequalities.

2.3 Generalized Temporal Bell Inequalities

In this section, we will generalize the two results (1) and (4) in such a way that more than two states are permitted. This will be of relevance when we discuss the possibility of "hidden mental states", i.e. of mental states which we need not be aware of.

We assume that the sets of possible results for the three observables 1, 2, and 3 are such that it is meaningful to say that the outcomes of measurement i and j are "equal" or "different". (Technically speaking, this is e.g. realized, if the spectra of the observables are equal). Again, the number $N^-(i,j)$ refers to the

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[4] Temporal Bell inequalities were initially discussed by Leggett et al. [5] and later by Mahler [6].

number of cases where the two observables i and j ($=$ 1,2 or 3) are in different states. Then the following inequality has to hold:

$$N^-(1,3) \leq N^-(1,2) + N^-(2,3)\,. \tag{5}$$

It just follows from the fact that if the results of measurements 1 and 3 are known to be different, then either the results of measurements 1 and 2 have to be different, or the results of measurements 2 and 3 (or both). This follows immediately from the transitivity of "being equal": If a and b are equal and b and c are equal then a and c have to be equal.

Inequality (5) is the same as for the case with only two possible measurement outcomes. Rewriting (5) as a temporal inequality and choosing the same t-values as in subsection 2.2 (and assuming time-translation invariance) we again obtain the sublinearity condition (4). Now, however, $N^-(\tau)$ refers to the number of states that are different at time t and time $t+\tau$.

Systems with an increasing number of states entail that temporal Bell inequalities are increasingly difficult to violate. However, the advantage of including more than two states is that the inequalities do not depend on the existence of "hidden states", i.e. states which one is unaware of. This will become relevant in our discussion of acategorial states below.

Note that (5) refers to a discrete number of states. The case of continuous variables is more difficult to deal with. Technically it is more difficult, because the condition of two states being equal is of measure zero. Practically it is more difficult, because the decision whether or not two states are equal is empirically more difficult to make.

3 Experimental Tests of Bell Inequalities in Mental Systems

For an experimental determination of $N^-(t_i,t_j)$ (or the corresponding probability) one might think of simply asking a subject about its mental state at time t_j and repeating the same question at time t_i. However, such observations clearly can have an influence on the mental state, so that they are typically invasive.

A similar situation occurs when individuals are asked to first memorize their mental states at time t_j and t_i and finally report whether or not the states were different. Even though in this case only one observation is made "externally", two "internal" self-observations must be made for the states to be memorized. Again, this form of self-observation may be invasive, and the later state may be different from the state which would have been realized if such a self-observation had not been made.

An example of a mental two-state system is given by the two percepts corresponding to the two possible perspectives of a Necker cube [8]. The Necker cube is a two-dimensional drawing of a cube whose perception is ambiguous with respect to its two perspectival interpretations.

Numerous publications report the distribution of dwell times for the two percepts, i.e. the probabilities $p^-(\tau)$ of perceiving different percepts at $t=0$ and

$t = \tau$). This distribution is well approximated by a gamma function (cf. Brascamp et al. [9]), which seems to indicate a violation of the temporal Bell inequality (4) in the regime of small times τ. However, these experiments are hardly non-invasive.

Even though we may never be able to fully guarantee that a measurement is non-invasive, an interesting candidate for such a measurement could be a scenario in which individuals are not asked to observe their states at t_i and t_j separately, but to judge at $t > (t_i, t_j)$ whether or not they have been the same at both instances. This would represent one single "product observation" with the possible results "same" or "different". The individual would not have to be aware of the states at time t_j and t_i separately. It is only necessary to report later whether or not they have been the same.

There is an analogue of this situation in quantum physics. In a double-slit experiment the particles may either show an interference pattern after the double slit or a single broad distribution. Any measured "which path"-information destroys the coherence between the two contributions of slit 1 and slit 2 and, therefore, the interference pattern. On the other hand, when "which path"-information is in principle unavailable, the interference patterns are observed. Similarly, the "which-state" information about the two states at time t_j and time t_i (corresponding to the "which path"-information in the double-slit scenario) destroys the non-classical behavior. Mere knowledge about whether or not the states were the same does not include "which-state" information.

As the result of such a single product measurement (same or not) depends on a correlation between states at different instances of time, we call such measurements "temporally non-local". They are also known in quantum theory, where they exhibit a somewhat tricky behavior though. Let us assume that a measurement is represented by the matrix σ_3, and the generator of the temporal evolution (the Hamiltonian H) by the matrix σ_1:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad H = g\sigma_1 = g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6}$$

Such a model was in fact investigated in the context of the bistable perception of ambiguous stimuli [10, 11], and it does indeed predict a violation of Bell inequalities under certain conditions [12, 7, 13].

The product operator $M(t) = \sigma_3(t)\sigma_3(0)$ has eigenvalues $\mathrm{e}^{\pm 2\mathrm{i}gt}$, where $1/g$ is the basic time scale of the evolution. These eigenvalues are not real – $M(t)$ is not a quantum observable because it is not self-adjoint. Self-adjoint combinations of $M(t)$ are the real part $S(t) = \frac{1}{2}(M(t) + M^{+}(t))$ with the single eigenvalue $\cos(2gt)$ and the imaginary part $A(t) = \frac{1}{2\mathrm{i}}(M(t) - M^{+}(t))$ with eigenvalues $\pm \sin 2gt$.

Interpreted as the possible outcomes of single measurements, these eigenvalues already show a non-classical effect. The eigenvalues of $\sigma(0)$ and $\sigma(t)$ are ± 1 each, so their product can only be $+1$ or -1. However, the eigenvalues of the product operators assume these values only for particular values of t. This indicates that non-deterministic quantum behavior is not the result of "hidden variables". In general, temporally non-local measurements are difficult to perform for quantum systems.

In the context of mental systems such measurements may be easier. As an example for a temporally non-local measurement we mention the determination of so-called order thresholds. It has been observed [14] that for time intervals between successive stimuli that are slightly below $\approx 30 - 70\,\mathrm{ms}$, individuals are able to distinguish the stimuli as not simultaneous without being able to assign their sequence correctly. Measurements of this type are interesting candidates for product measurements with different values of time intervals τ in order to test the sublinearity condition for $p^-(\tau)$ and $p^-(2\tau)$.

Finally, we should like to mention that models with non-commuting observables can easily be found in the classical realm. Simple examples are neural networks, where the presentation of an input pattern may be interpreted as an observation and the measurement of the reaction of the network at the output nodes as the result of this observation.[5] One can easily construct examples of this type which seem to violate temporal Bell inequalities. However, the application of an input pattern is an invasive operation.

4 Conclusions

We derived an inequality for correlations between the results of observations, which can be interpreted as a temporal Bell inequality and which has to hold under the assumption that the state of a system determines the outcome of any measurement among a class of (non-commuting) observables. This temporal inequality can be tested for mental systems as well. However, the main challenge is that measurements be non-invasive, i.e., the result of a second measurement assumed to be determined by the initial state is not changed due to the first measurement. We discussed temporally non-local measurements as a possibility to circumvent the difficulties related to this challenge.

Non-commuting observables can be implemented in very simple, strictly deterministic systems, where they do not lead to a violation of Bell inequalities. Neural networks provide a simple example for *invasive* measurements which lead to a violation of Bell inequalities. Once more, this highlights the issue of non-invasive measurements.

The extension of Bell inequalities to systems with more than two states may prove to be relevant for non-classical mental states in the sense of acategorial states [16, 17] (this term was first introduced by Jean Gebser [18]). While non-classical states of quantum systems may be interpreted as superpositions of states with classical properties, the more general notion of an acategorial state is particular suited for mental systems.

Such acategorial states may refer to transition phases between common categorial representations, and it has been proposed that they represent examples for states with "non-conceptual content" [17]. In analogy with quantum theory,

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[5] For more details see [15] where non-commutating observables were investigated for a special class of neural networks.

any attempt to direct one's conscious attention to an acategorial state corresponds to a measurement and destroys the state. We may speculate that the the decision of whether or not two successive states are the same or not, as an example for a temporally non-local, non-invasive measurement, may be suitable to provide indirect evidence for acategorial states without directly probing and thus destroying them.

Two directions of further research along the lines discussed seem to be promising: (1) performing experiments with temporally non-local, non-invasive measurements, and (2) testing Bell inequalities in recurrent and non-deterministic generalizations of neural network models.

Acknowledgement

We acknowledge discussions with P. beim Graben and H. Römer. One of us (T.F.) acknowledges partial funding of this research by the Fetzer Foundation Trust.

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